

EUCLID'S GEOMETRY

The postulate on parallels . . . was in antiquity the final solution of a problem that must have preoccupied Greek mathematics for a long period before Euclid.

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THE ORIGINS OF GEOMETRY

The word “geometry” comes from the Greek *geometrein* (*geo-*, “earth,” and *metrein*, “to measure”); geometry was originally the science of measuring land. The Greek historian Herodotus (5th century B.C.) credits Egyptian surveyors with having originated the subject of geometry, but other ancient civilizations (Babylonian, Hindu, Chinese) also possessed much geometric information.

Ancient geometry was actually a collection of rule-of-thumb procedures arrived at through experimentation, observation of analogies, guessing, and occasional flashes of intuition. In short, it was an empirical subject in which approximate answers were usually sufficient for practical purposes. The Babylonians of 2000 to 1600 B.C. considered the circumference of a circle to be three times the diameter; i.e., they took π to be equal to 3. This was the value given by the Roman architect Vitruvius and it is found in the Chinese literature as well. It was even considered sacred by the ancient Jews and sanctioned in scripture (I Kings 7:23) — an attempt by Rabbi Nehemiah to change

the value of π to $\frac{22}{7}$ was rejected. The Egyptians of 1800 B.C., according to the Rhind papyrus, had the approximation $\pi \sim (\frac{16}{9})^2 \sim 3.1604$.¹

Sometimes the Egyptians guessed correctly, other times not. They found the correct formula for the volume of a frustum of a square pyramid—a remarkable accomplishment. On the other hand, they thought that a formula for area that was correct for rectangles applied to *any* quadrilateral. Egyptian geometry was not a science in the Greek sense, only a grab bag of rules for calculation without any motivation or justification.

The Babylonians were much more advanced than the Egyptians in arithmetic and algebra. Moreover, they knew the Pythagorean theorem—in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs—long before Pythagoras was born. Recent research by Otto Neugebauer has revealed the heretofore unknown Babylonian algebraic influence on Greek mathematics.

However, the Greeks, beginning with Thales of Miletus, insisted that geometric statements be established by deductive reasoning rather than by trial and error. Thales was familiar with the computations, partly right and partly wrong, handed down from Egyptian and Babylonian mathematics. In determining which results were correct, he developed the first logical geometry (Thales is also famous for having predicted the eclipse of the sun in 585 B.C.). The orderly development of theorems by *proof* was characteristic of Greek mathematics and entirely new.

The systematization begun by Thales was continued over the next two centuries by Pythagoras and his disciples. Pythagoras was regarded by his contemporaries as a religious prophet. He preached the immortality of the soul and reincarnation. He organized a brotherhood of believers that had its own purification and initiation rites, followed a vegetarian diet, and shared all property communally. The Pythagoreans differed from other religious sects in their belief that elevation of

¹ In recent years π has been approximated to a very large number of decimal places by computers; to five places, π is approximately 3.14159. In 1789 Johann Lambert proved that π was not equal to any fraction (rational number), and in 1882 F. Lindemann proved that π is a *transcendental number*, in the sense that it does not satisfy any algebraic equation with rational coefficients, which implies that in the Euclidean plane, it is impossible to square a circle using only straightedge and compass.

the soul and union with God are achieved by the study of music and mathematics. In music, Pythagoras calculated the correct ratios of the harmonic intervals. In mathematics, he taught the mysterious and wonderful properties of numbers. Book VII of Euclid's *Elements* is the text of the theory of numbers taught in the Pythagorean school.

The Pythagoreans were greatly shocked when they discovered irrational lengths, such as $\sqrt{2}$ (see Chapter 2, pp. 43–44). At first they tried to keep this discovery secret. The historian Proclus wrote: "It is well known that the man who first made public the theory of irrationals perished in a shipwreck, in order that the inexpressible and unimaginable should ever remain veiled." Since the Pythagoreans did not consider $\sqrt{2}$ a number, they transmuted their algebra into geometric form in order to represent $\sqrt{2}$ and other irrational lengths by segments ($\sqrt{2}$ by a diagonal of the unit square).

The systematic foundation of plane geometry by the Pythagorean school was brought to a conclusion around 400 B.C. in the *Elements* by the mathematician Hippocrates (not to be confused with the physician of the same name). Although this treatise has been lost, we can safely say that it covered most of Books I–IV of Euclid's *Elements*, which appeared about a century later. The Pythagoreans were never able to develop a theory of proportions that was also valid for irrational lengths. This was later achieved by Eudoxus, whose theory was incorporated into Book V of Euclid's *Elements*.

The fourth century B.C. saw the flourishing of Plato's Academy of science and philosophy (founded about 387 B.C.). In the *Republic* Plato wrote, "The study of mathematics develops and sets into operation a mental organism more valuable than a thousand eyes, because through it alone can truth be apprehended." Plato taught that the universe of ideas is more important than the material world of the senses, the latter being only a shadow of the former. The material world is an unlit cave on whose walls we see only shadows of the real, sunlit world outside. The errors of the senses must be corrected by concentrated thought, which is best learned by studying mathematics. The Socratic method of dialog is essentially that of indirect proof, by which an assertion is shown to be invalid if it leads to a contradiction. Plato repeatedly cited the proof for the irrationality of the length of a diagonal of the unit square as an illustration of the method of indirect proof (the *reductio ad absurdum*—see Chapter 2, pp. 42–44). The point is that this irrationality of length could never have been discov-

ered by physical measurements, which always include a small experimental margin of error.

Euclid was a disciple of the Platonic school. Around 300 B.C. he produced the definitive treatment of Greek geometry and number theory in his 13-volume *Elements*. In compiling this masterpiece Euclid built on the experience and achievements of his predecessors in preceding centuries: on the Pythagoreans for Books I–IV, VII, and IX, Archytas for Book VIII, Eudoxus for Books V, VI, and XII, and Theaetetus for Books X and XIII. So completely did Euclid's work supersede earlier attempts at presenting geometry that few traces remain of these efforts. It's a pity that Euclid's heirs have not been able to collect royalties on his work, for he is the most widely read author in the history of mankind. His approach to geometry has dominated the teaching of the subject for over two thousand years. Moreover, the axiomatic method used by Euclid is the prototype for all of what we now call "pure mathematics." It is pure in the sense of "pure thought": no physical experiments need be performed to verify that the statements are correct — only the reasoning in the demonstrations need be checked.

Euclid's *Elements* is pure also in that the work includes no practical applications. Of course, Euclid's geometry has had an enormous number of applications to practical problems in engineering, but they are not mentioned in the *Elements*. According to legend, a beginning student of geometry asked Euclid, "What shall I get by learning these things?" Euclid called his slave, saying, "Give him a coin, since he must make gain out of what he learns." To this day, this attitude toward application persists among many pure mathematicians — they study mathematics for its own sake, for its intrinsic beauty and elegance (see essay topics 5 and 8 in Chapter 8).

Surprisingly enough, as we will see later, pure mathematics often turns out to have applications never dreamt of by its creators — the "impractical" outlook of pure mathematicians is ultimately useful to society. Moreover, those parts of mathematics that have not been "applied" are also valuable to society, either as aesthetic works comparable to music and art or as contributions to the expansion of human consciousness and understanding.²

² For more detailed information on ancient mathematics, see Bartel van der Waerden (1961).

THE AXIOMATIC METHOD

Mathematicians can make use of trial and error, computation of special cases, inspired guessing, or any other way to discover theorems. The axiomatic method is a method of *proving* that results are correct. Some of the most important results in mathematics were originally given only incomplete proofs (we shall see that even Euclid was guilty of this). No matter—correct proofs would be supplied later (sometimes much later) and the mathematical world would be satisfied.

So proofs give us assurance that results are correct. In many cases they also give us more *general* results. For example, the Egyptians and Hindus knew by experiment that if a triangle has sides of lengths 3, 4, and 5, it is a right triangle. But the Greeks proved that if a triangle has sides of lengths a , b , and c and if $a^2 + b^2 = c^2$, then the triangle is a right triangle. It would take an infinite number of experiments to check this result (and, besides, experiments only measure things approximately). Finally, proofs give us tremendous insight into relationships among different things we are studying, forcing us to organize our ideas in a coherent way. You will appreciate this by the end of Chapter 6 (if not sooner).

What is the axiomatic method? If I wish to persuade you by *pure reasoning* to believe some statement S_1 , I could show you how this statement follows logically from some other statement S_2 that you may already accept. However, if you don't believe S_2 , I would have to show you how S_2 follows logically from some other statement S_3 . I might have to repeat this procedure several times until I reach some statement that you already accept, one I do not need to justify. That statement plays the role of an *axiom* (or *postulate*). If I cannot reach a statement that you will accept as the basis of my argument, I will be caught in an "infinite regress," giving one demonstration after another without end.

So there are two requirements that must be met for us to agree that a proof is correct:

REQUIREMENT 1. Acceptance of certain statements called "axioms," or "postulates," without further justification.

REQUIREMENT 2. Agreement on how and when one statement "fol-

lows logically” from another, i.e., agreement on certain rules of reasoning.

Euclid’s monumental achievement was to single out a few simple postulates, statements that were acceptable without further justification, and then to deduce from them 465 propositions, many complicated and not at all intuitively obvious, which contained all the geometric knowledge of his time. One reason the *Elements* is such a beautiful work is that so much has been deduced from so little.

UNDEFINED TERMS

We have been discussing what is required for us to agree that a proof is correct. Here is one requirement that we took for granted:

REQUIREMENT 0. Mutual understanding of the meaning of the words and symbols used in the discourse.

There should be no problem in reaching mutual understanding so long as we use terms familiar to both of us and use them consistently. If I use an unfamiliar term, you have the right to demand a *definition* of this term. Definitions cannot be given arbitrarily; they are subject to the rules of reasoning referred to (but not specified) in Requirement 2. If, for example, I define a right angle to be a 90° angle, and then define a 90° angle to be a right angle, I would violate the rule against *circular reasoning*.

Also, we cannot define every term that we use. In order to define one term we must use other terms, and to define these terms we must use still other terms, and so on. If we were not allowed to leave some terms *undefined*, we would get involved in infinite regress.

Euclid did attempt to define all geometric terms. He defined a “straight line” to be “that which lies evenly with the points on itself.” This definition is not very useful; to understand it you must already have the image of a line. So it is better to take “line” as an undefined term. Similarly, Euclid defined a “point” as “that which has no part” —again, not very informative. So we will also accept “point” as an undefined term. Here are the five undefined geometric terms that are

the basis for defining all other geometric terms in plane Euclidean geometry:

point

line

lie on (as in “two points *lie on* a unique line”)

between (as in “point C is *between* points A and B”)

congruent

For solid geometry, we would have to introduce a further undefined geometric term, “plane,” and extend the relation “lie on” to allow points and lines to lie on planes. *In this book (unless otherwise stated) we will restrict our attention to plane geometry, i.e., to one single plane.* So we define *the plane* to be the set of all points and lines, all of which are said to “lie on” it.

There are expressions that are often used synonymously with “lie on.” Instead of saying “point P *lies on* line l ,” we sometimes say “ l *passes through* P” or “P is *incident with* l ,” denoted $P \mathcal{I} l$. If point P lies on both line l and line m , we say that “ l and m *have point P in common*” or that “ l and m *intersect (or meet) in the point P.*”

The second undefined term, “line,” is synonymous with “straight line.” The adjective “straight” is confusing when it modifies the noun “line,” so we won’t use it. Nor will we talk about “curved lines.” Although the word “line” will not be defined, its use will be restricted by the axioms for our geometry. For instance, one axiom states that two given points lie on only one line. Thus, in Figure 1.1, l and m could not both represent lines in our geometry, since they both pass through the points P and Q.

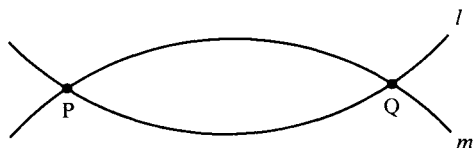


FIGURE 1.1

There are other mathematical terms that we will use that should be added to our list of undefined terms, since we won’t define them; they have been omitted because they are not specifically geometric in nature, but are rather what Euclid called “common notions.” Nevertheless, since there may be some confusion about these terms, a few remarks are in order.

The word "set" is fundamental in all of mathematics today; it is now used in elementary schools, so undoubtedly you are familiar with its use. Think of it as a "collection of objects." Two related notions are "belonging to" a set or "being an element (or member) of" a set, as in our convention that all points and lines *belong* to the plane. If every element of a set S is also an element of a set T , we say that S is "contained in" or "part of" or "a subset of" T . We will define "segment," "ray," "circle," and other geometric terms to be certain sets of points. A "line," however, is not a set of points in our treatment (for reasons of duality in Chapter 2). When we need to refer to the set of all points lying on a line l , we will denote that set by $\{l\}$.

In the language of sets we say that sets S and T are *equal* if every member of S is a member of T , and vice versa. For example, the set S of all authors of Euclid's *Elements* is (presumably) equal to the set whose only member is Euclid. Thus, "equal" means "identical."

Euclid used the word "equal" in a different sense, as in his assertion that "base angles of an isosceles triangle are *equal*." He meant that base angles of an isosceles triangle have an equal number of degrees, not that they are identical angles. So to avoid confusion we will not use the word "equal" in Euclid's sense. Instead, we will use the undefined term "congruent" and say that "base angles of an isosceles triangle are *congruent*." Similarly, we don't say that "if AB *equals* AC , then $\triangle ABC$ is isosceles." (If AB *equals* AC , following our use of the word "equals," $\triangle ABC$ is not a triangle at all, only a segment.) Instead, we would say that "if AB is *congruent* to AC , then $\triangle ABC$ is *isosceles*." This use of the undefined term "congruent" is more general than the one to which you are accustomed; it applies not only to triangles but to angles and segments as well. To understand the use of this word, picture congruent objects as "having the same size and shape."

Of course, we must specify (as Euclid did in his "common notions") that "a thing is congruent to itself," and that "things congruent to the same thing are congruent to each other." Statements like these will later be included among our axioms of congruence (Chapter 3).

The list of undefined geometric terms shown earlier in this section is due to David Hilbert (1862–1943). His treatise *The Foundations of Geometry* (1899) not only clarified Euclid's definitions but also filled in the gaps in some of Euclid's proofs. Hilbert recognized that Euclid's proof for the side-angle-side criterion of congruence in triangles was based on an unstated assumption (the principle of superposition), and that this criterion had to be treated as an axiom. He also built on the

earlier work of Moritz Pasch, who in 1882 published the first rigorous treatise on geometry; Pasch made explicit Euclid's unstated assumptions about betweenness (the axioms on betweenness will be studied in Chapter 3). Some other mathematicians who worked to establish rigorous foundations for Euclidean geometry are: G. Peano, M. Pieri, G. Veronese, O. Veblen, G. de B. Robinson, E. V. Huntington, and H. G. Forder. These mathematicians used lists of undefined terms different from the one used by Hilbert. Pieri used only two undefined terms (as a result, however, his axioms were more complicated). The selection of undefined terms and axioms is *arbitrary*; Hilbert's selection is popular because it leads to an elegant development of geometry similar to Euclid's presentation.

EUCLID'S FIRST FOUR POSTULATES

Euclid based his geometry on five fundamental assumptions, called *axioms* or *postulates*.

EUCLID'S POSTULATE I. For every point P and for every point Q not equal to P there exists a unique line l that passes through P and Q .

This postulate is sometimes expressed informally by saying "two points determine a unique line." We will denote the unique line that passes through P and Q by \overleftrightarrow{PQ} .

To state the second postulate, we must make our first definition.

DEFINITION. Given two points A and B . The *segment* AB is the set whose members are the points A and B and all points that lie on the line \overleftrightarrow{AB} and are between A and B (Figure 1.2). The two given points A and B are called the *endpoints* of the segment AB .³

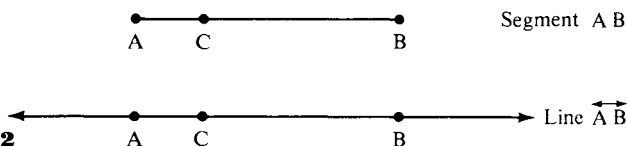


FIGURE 1.2

³ Warning on notation: In many high school geometry texts the notation \overline{AB} is used for "segment AB ."

EUCLID'S POSTULATE II. For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE (Figure 1.3).

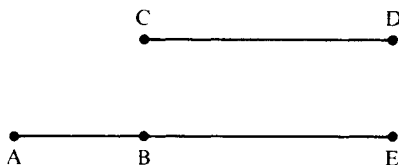


FIGURE 1.3 $CD \cong BE$.

This postulate is sometimes expressed informally by saying that “any segment AB can be extended by a segment BE congruent to a given segment CD .” Notice that in this postulate we have used the undefined term “congruent” in the new way, and we use the usual notation $CD \cong BE$ to express the fact that CD is congruent to BE .

In order to state the third postulate, we must introduce another definition.

DEFINITION. Given two points O and A . The set of all points P such that segment OP is congruent to segment OA is called a *circle* with O as *center*, and each of the segments OP is called a *radius* of the circle.

It follows from Euclid's previously mentioned common notion (“a thing is congruent to itself”) that $OA \cong OA$, so A is also a point on the circle just defined.

EUCLID'S POSTULATE III. For every point O and every point A not equal to O there exists a circle with center O and radius OA (Figure 1.4).

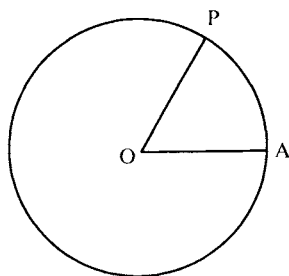


FIGURE 1.4 Circle with center O and radius OA .

Actually, because we are using the language of sets rather than that of Euclid, it is not really necessary to assume this postulate; it is a consequence of set theory that the set of all points P with $OP \cong OA$ exists. Euclid had in mind *drawing* the circle with center O and radius OA , and this postulate tells you that such a drawing is allowed, for example, with a compass. Similarly, in Postulate II you are allowed to extend segment AB by drawing segment BE with a straightedge. Our treatment “purifies” Euclid by eliminating references to drawing in our proofs.⁴ But you should review the straightedge and compass constructions in Major Exercise 1.

DEFINITION. The ray \overrightarrow{AB} is the following set of points lying on the line \overleftrightarrow{AB} : those points that belong to the segment \overline{AB} and all points C on \overleftrightarrow{AB} such that B is between A and C . The ray \overrightarrow{AB} is said to *emanate from the vertex* A and to be *part of line* \overleftrightarrow{AB} . (See Figure 1.5.)

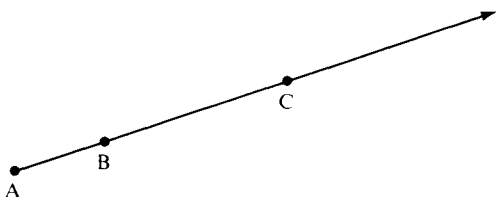


FIGURE 1.5 Ray \overrightarrow{AB} .

DEFINITION. Rays \overrightarrow{AB} and \overrightarrow{AC} are *opposite* if they are distinct, if they emanate from the same point A , and if they are part of the same line $\overleftrightarrow{AB} = \overleftrightarrow{AC}$ (Figure 1.6.).

⁴ However, it is a fascinating mathematical problem to determine just what geometric constructions are possible using only a compass and straightedge. Not until the nineteenth century was it proved that such constructions as trisecting an arbitrary angle, squaring a circle, or doubling a cube were impossible using only a compass and straightedge. Pierre Wantzel proved this by translating the geometric problem into an algebraic problem; he showed that straightedge and compass constructions correspond to a solution of certain algebraic equations using only the operations of addition, subtraction, multiplication, division, and extraction of square roots. For the particular algebraic equations obtained from, say, the problem of trisecting an arbitrary angle, such a solution is impossible because cube roots are needed. Of course, it is possible to trisect angles using other instruments, such as a marked straightedge and compass (see Major Exercise 3 and Projects 1, 2, and 4), and J. Bolyai proved that in the hyperbolic plane, it is possible to “square” the circle.

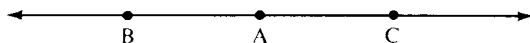


FIGURE 1.6 Opposite rays.

DEFINITION. An “*angle with vertex A*” is a point A together with two distinct nonopposite rays \overrightarrow{AB} and \overrightarrow{AC} (called the *sides* of the angle) emanating from A.⁵ (See Figure 1.7.)

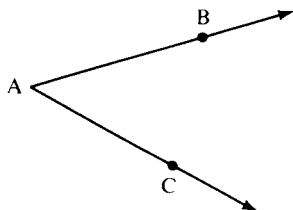


FIGURE 1.7 Angle with vertex A.

We use the notations $\sphericalangle A$, $\sphericalangle BAC$, or $\sphericalangle CAB$ for this angle.

DEFINITION. If two angles $\sphericalangle BAD$ and $\sphericalangle CAD$ have a common side \overrightarrow{AD} and the other two sides \overrightarrow{AB} and \overrightarrow{AC} form opposite rays, the angles are *supplements* of each other, or *supplementary angles* (Figure 1.8).

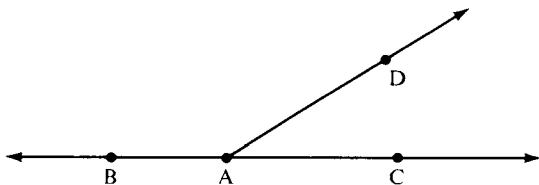


FIGURE 1.8 Supplementary angles.

DEFINITION. An angle $\sphericalangle BAD$ is a *right angle* if it has a supplementary angle to which it is congruent (Figure 1.9).

⁵ According to this definition, there is no such thing as a “straight angle.” We eliminated this expression because most of the assertions we will make about angles do not apply to “straight angles.” The definition excludes zero angles as well.

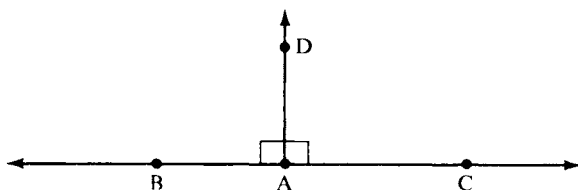


FIGURE 1.9 Right angles $\angle BAD \cong \angle CAD$.

We have thus succeeded in defining a right angle without referring to “degrees,” by using the undefined notion of congruence of angles. “Degrees” will not be introduced formally until Chapter 4, although we will occasionally refer to them in informal discussions.

We can now state Euclid’s fourth postulate.

EUCLID’S POSTULATE IV. All right angles are congruent to each other.

This postulate expresses a sort of homogeneity; even though two right angles may be “very far away” from each other, they nevertheless “have the same size.” The postulate therefore provides a natural standard of measurement for angles.⁶

THE PARALLEL POSTULATE

Euclid’s first four postulates have always been readily accepted by mathematicians. The fifth (parallel) postulate, however, was highly controversial. In fact, as we shall see later, consideration of alternatives to Euclid’s parallel postulate resulted in the development of non-Euclidean geometries.

At this point we are not going to state the fifth postulate in its original form, as it appeared in the *Elements*. Instead, we will present a simpler postulate (which we will later show is logically equivalent to Euclid’s original). This version is sometimes called *Playfair’s postulate*

⁶ On the contrary, there is no natural standard of measurement for *lengths* in Euclidean geometry. Units of length (one foot, one meter, etc.) must be chosen arbitrarily. The remarkable fact about hyperbolic geometry, on the other hand, is that it does admit a natural standard of length (see Chapter 6).

because it appeared in John Playfair's presentation of Euclidean geometry, published in 1795 — although it was referred to much earlier by Proclus (A.D. 410–485). We will call it *the Euclidean parallel postulate* because it distinguishes Euclidean geometry from other geometries based on parallel postulates. The most important definition in this book is the following:

DEFINITION. Two lines l and m are *parallel* if they do not intersect, i.e., if no point lies on both of them. We denote this by $l \parallel m$.

Notice first that we assume the lines lie in the same plane (because of our convention that all points and lines lie in one plane, unless stated otherwise; in space there are noncoplanar lines which fail to intersect and they are called *skew lines*, not “parallel”). Notice secondly what the definition does *not* say: it does not say that the lines are equidistant, i.e., it does not say that the distance between the two lines is everywhere the same. Don't be misled by drawings of parallel lines in which the lines appear to be equidistant. We want to be rigorous here and so should not introduce into our proofs assumptions that have not been stated explicitly. At the same time, don't jump to the conclusion that parallel lines are *not* equidistant. We are not committing ourselves either way and shall reserve judgment until we study the matter further. At this point, the only thing we know for sure about parallel lines is that they do not meet.

THE EUCLIDEAN PARALLEL POSTULATE. For every line l and for every point P that does not lie on l there exists a unique line m through P that is parallel to l . (See Figure 1.10.)

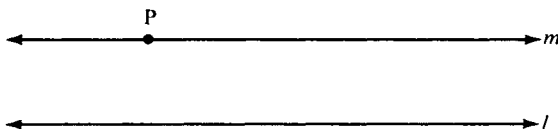


FIGURE 1.10 Lines l and m are parallel.

Why should this postulate be so controversial? It may seem “obvious” to you, perhaps because you have been conditioned to think in Euclidean terms. However, if we consider the axioms of geometry as abstractions from experience, we can see a difference between this

postulate and the other four. The first two postulates are abstractions from our experiences drawing with a straightedge; the third postulate derives from our experiences drawing with a compass. The fourth postulate is perhaps less obvious as an abstraction; nevertheless it derives from our experiences measuring angles with a protractor (where the sum of supplementary angles is 180° , so that if supplementary angles are congruent to each other, they must each measure 90°).

The fifth postulate is different in that we cannot verify empirically whether two lines meet, since we can draw only segments, not lines. We can extend the segments further and further to see if they meet, but we cannot go on extending them forever. Our only recourse is to verify parallelism indirectly, by using criteria other than the definition.

What is another criterion for l to be parallel to m ? Euclid suggested drawing a *transversal* (i.e., a line t that intersects both l and m in distinct points), and measuring the number of degrees in the interior angles α and β on one side of t . Euclid predicted that if the sum of angles α and β turns out to be less than 180° , the lines (if produced sufficiently far) would meet on the same side of t as angles α and β (see Figure 1.11). This, in fact, is the content of Euclid's fifth postulate.

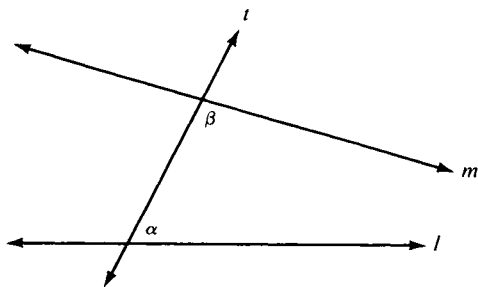


FIGURE 1.11

The trouble with this criterion for parallelism is that it turns out to be logically equivalent to the Euclidean parallel postulate that was just stated (see the section *Equivalence of Parallel Postulates* in Chapter 4.). So we cannot use this criterion to convince ourselves of the correctness of the parallel postulate — that would be circular reasoning. Euclid himself recognized the questionable nature of the parallel postulate, for he postponed using it for as long as he could (until the proof of his 29th proposition).

ATTEMPTS TO PROVE THE PARALLEL POSTULATE

Remember that an axiom was originally supposed to be so simple and intuitively obvious that no one could doubt its validity. From the very beginning, however, the parallel postulate was attacked as insufficiently plausible to qualify as an unproved assumption. For two thousand years mathematicians tried to derive it from the other four postulates or to replace it with another postulate, one more self-evident. All attempts to derive it from the first four postulates turned out to be unsuccessful because the so-called proofs always entailed a hidden assumption that was unjustifiable. The substitute postulates, purportedly more self-evident, turned out to be logically equivalent to the parallel postulate, so that nothing was gained by the substitution. We will examine these attempts in detail in Chapter 5, for they are very instructive. For the moment, let us consider one such effort.

The Frenchman Adrien Marie Legendre (1752–1833) was one of the best mathematicians of his time, contributing important discoveries to many different branches of mathematics. Yet he was so obsessed with proving the parallel postulate that over a period of 29 years he published one attempt after another in different editions of his *Éléments de Géométrie*.⁷ Here is one attempt (see Figure 1.12):

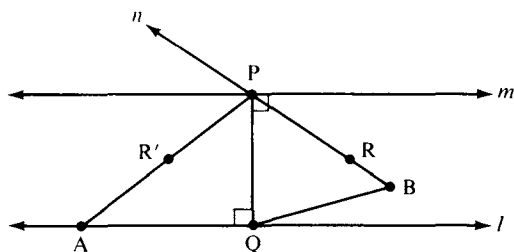
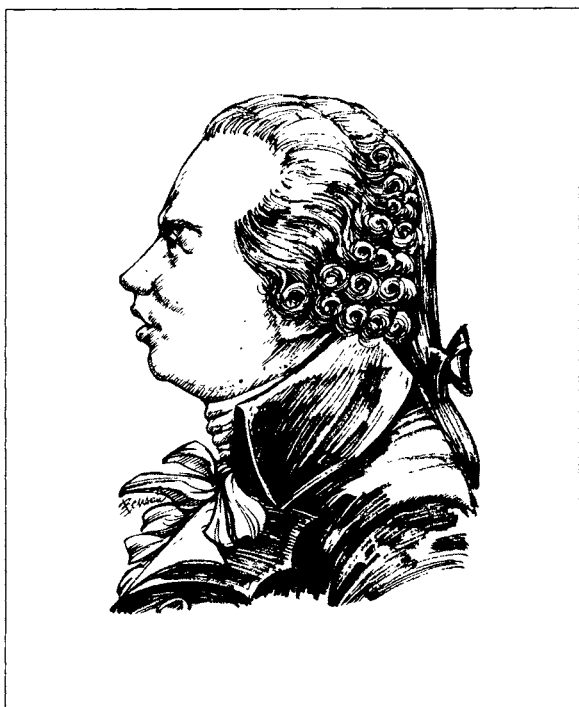


FIGURE 1.12

Given P not on line l . Drop perpendicular PQ from P to l at Q . Let m be the line through P perpendicular to \overrightarrow{PQ} . Then m is parallel to l ,

⁷ Davies' translation of the *Éléments* was the most popular geometry textbook in the United States during the nineteenth century. Legendre is best known for the method of least squares in statistics, the law of quadratic reciprocity in number theory, and the Legendre polynomials in differential equations. His attempts to prove the parallel postulate led to two important theorems in neutral geometry (see Chapter 4).



Adrien Marie Legendre

since l and m have the common perpendicular \overleftrightarrow{PQ} . Let n be any line through P distinct from m and \overleftrightarrow{PQ} . We must show that n meets l . Let \overrightarrow{PR} be a ray of n between \overleftrightarrow{PQ} and a ray of m emanating from P . There is a point R' on the opposite side of \overleftrightarrow{PQ} from R such that $\angle QPR' \cong \angle QPR$. Then Q lies in the interior of $\angle RPR'$. Since line l passes through the point Q interior to $\angle RPR'$, l must intersect one of the sides of this angle. If l meets side \overrightarrow{PR} , then certainly l meets n . Suppose l meets side $\overrightarrow{PR'}$ at a point A . Let B be the unique point on side \overrightarrow{PR} such that $PA \cong PB$. Then $\triangle PQA \cong \triangle PQB$ (SAS); hence $\angle PQB$ is a right angle, so that B lies on l (and n).

You may feel that this argument is plausible enough. Yet how could you tell if it is correct? You would have to justify each step, first defining each term carefully. For instance, you would have to define what was meant by two lines being "perpendicular"—otherwise,

how could you justify the assertion that lines l and m are parallel simply because they have a common perpendicular? (You would first have to prove that as a separate theorem, if you could.) You would have to justify the side-angle-side (SAS) criterion of congruence in the last statement. You would have to define the “interior” of an angle, and prove that a line through the interior of an angle must intersect one of the sides. In proving all of these things, you would have to be sure to use only the first four postulates and not any statement equivalent to the fifth; otherwise the argument would be circular.

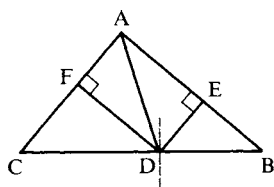
Thus there is a lot of work that must be done before we can detect the flaw. In the next few chapters we will do this preparatory work so that we can confidently decide whether or not Legendre’s proposed proof is valid. (Legendre’s argument contains several statements that cannot be proved from the first four postulates.) As a result of this work we will be better able to understand the foundations of Euclidean geometry. We will discover that a large part of this geometry is independent of the theory of parallels and is equally valid in hyperbolic geometry.

THE DANGER IN DIAGRAMS

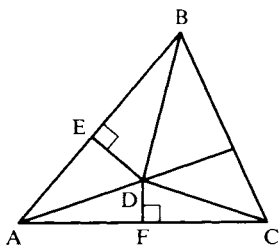
Diagrams have always been helpful in understanding geometry—they are included in Euclid’s *Elements* and they are included in this book. But there is a danger that a diagram may suggest a fallacious argument. A diagram may be slightly inaccurate or it may represent only a special case. If we are to recognize the flaws in arguments such as Legendre’s, we must not be misled by diagrams that *look* plausible.

What follows is a well-known and rather involved argument that pretends to prove that all triangles are isosceles. Place yourself in the context of what you know from high school geometry. (After this chapter you will have to put that knowledge on hold.) Find the flaw in the argument.

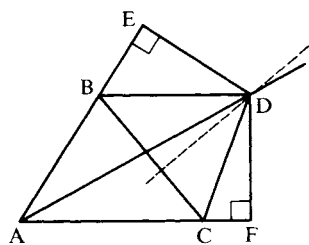
Given $\triangle ABC$. Construct the bisector of $\angle A$ and the perpendicular bisector of side BC opposite to $\angle A$. Consider the various cases (Figure 1.13).



Case 3



Case 2



Case 4

FIGURE 1.13

Case 1. The bisector of $\angle A$ and the perpendicular bisector of segment BC are either parallel or identical. In either case, the bisector of $\angle A$ is perpendicular to BC and hence, by definition, is an altitude. Therefore, the triangle is isosceles. (The conclusion follows from the Euclidean theorem: if an angle bisector and altitude from the same vertex of a triangle coincide, the triangle is isosceles.)

Suppose now that the bisector of $\angle A$ and the perpendicular bisector of the side opposite are not parallel and do not coincide. Then they intersect in exactly one point, D , and there are three cases to consider:

Case 2. The point D is inside the triangle.

Case 3. The point D is on the triangle.

Case 4. The point D is outside the triangle.

For each case construct DE perpendicular to AB and DF perpendicular to AC , and for cases 2 and 4 join D to B and D to C . In each case, the following proof now holds (see Figure 1.13):

$DE \cong DF$ because all points on an angle bisector are equidistant from the sides of the angle; $DA \cong DA$, and $\angle DEA$ and $\angle DFA$ are right angles; hence, $\triangle ADE$ is congruent to $\triangle ADF$ by the hypotenuse-leg theorem of Euclidean geometry. (We could also have used the SAA theorem with $DA \cong DA$, and the bisected angle and right angles.) Therefore, we have $AE \cong AF$. Now, $DB \cong DC$ because all points on the perpendicular bisector of a segment are equidistant from the ends of the segment. Also, $DE \cong DF$, and $\angle DEB$ and $\angle DFC$ are right angles. Hence, $\triangle DEB$ is congruent to $\triangle DFC$ by the hypotenuse-leg theorem, and hence $FC \cong BE$. It follows that $AB \cong AC$ —in cases 2 and 3 by addition and in case 4 by subtraction. The triangle is therefore isosceles.

THE POWER OF DIAGRAMS

Geometry, for human beings (perhaps not for computers), is a visual subject. Correct diagrams are extremely helpful in understanding proofs and in discovering new results. One of the best illustrations of this is Figure 1.14, which reveals immediately the validity of the

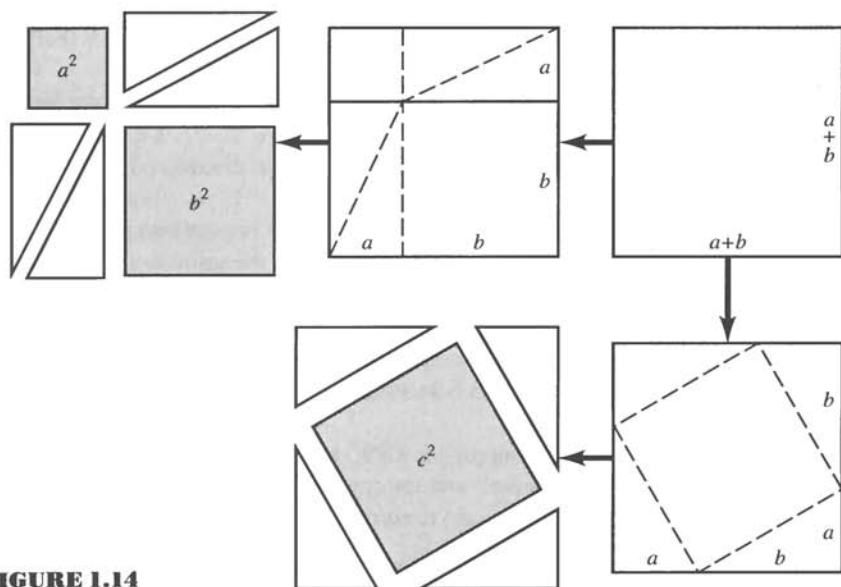
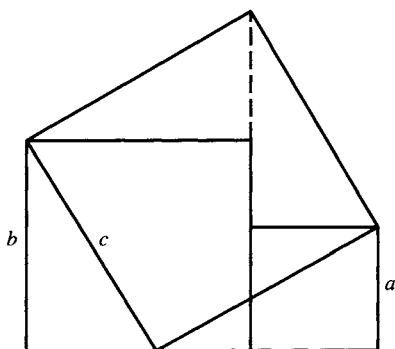


FIGURE 1.14

**FIGURE 1.15**

Pythagorean theorem in Euclidean geometry. (Euclid's proof was much more complicated.) Figure 1.15 is a simpler diagram suggesting a proof by dissection.

REVIEW EXERCISE

Which of the following statements are correct?

- (1) The Euclidean parallel postulate states that for every line l and for every point P not lying on l there exists a unique line m through P that is parallel to l .
- (2) An "angle" is defined as the space between two rays that emanate from a common point.
- (3) Most of the results in Euclid's *Elements* were discovered by Euclid himself.
- (4) By definition, a line m is "parallel" to a line l if for any two points P, Q on m , the perpendicular distance from P to l is the same as the perpendicular distance from Q to l .
- (5) It was unnecessary for Euclid to assume the parallel postulate because the French mathematician Legendre proved it.
- (6) A "transversal" to two lines is another line that intersects both of them in distinct points.
- (7) By definition, a "right angle" is a 90° angle.
- (8) "Axioms" or "postulates" are statements that are assumed, without further justification, whereas "theorems" or "propositions" are proved using the axioms.

- (9) We call $\sqrt{2}$ an “irrational number” because it cannot be expressed as a quotient of two whole numbers.
- (10) The ancient Greeks were the first to insist on proofs for mathematical statements to make sure they were correct.

EXERCISES

In Exercises 1–4 you are asked to define some familiar geometric terms. The exercises provide a review of these terms as well as practice in formulating definitions with precision. In making a definition, you may use the five undefined geometric terms and all other geometric terms that have been defined in the text so far or in any preceding exercises.

Making a definition sometimes requires a bit of thought. For example, how would you define *perpendicularity* for two lines l and m ? A first attempt might be to say that “ l and m intersect and at their point of intersection these lines form right angles.” It would be legitimate to use the terms “intersect” and “right angle” because they have been previously defined. But what is meant by the statement that *lines* form right angles? Surely, we can all draw a picture to show what we mean, but the problem is to express the idea verbally, using only terms introduced previously. According to the definition on p. 17, an angle is formed by two nonopposite *rays* emanating from the same vertex. We may therefore define l and m as *perpendicular* if they intersect at a point A and if there is a ray \overrightarrow{AB} that is part of l and a ray \overrightarrow{AC} that is part of m such that $\angle BAC$ is a right angle (Figure 1.16). We denote this by $l \perp m$.

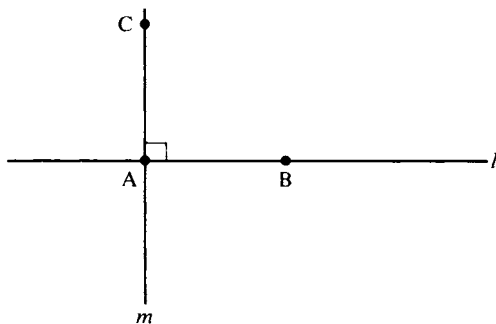


FIGURE 1.16 Perpendicular lines.

1. Define the following terms:

- Midpoint* M of a segment AB.
- Perpendicular bisector* of a segment AB (you may use the term “midpoint” since you have just defined it).
- Ray \overrightarrow{BD} *bisects* angle $\angle ABC$ (given that point D is between A and C).
- Points A, B, and C are *collinear*.
- Lines l , m , and n are *concurrent* (see Figure 1.17).

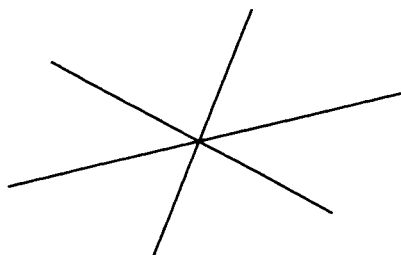


FIGURE 1.17 Concurrent lines.

2. Define the following terms:

- The *triangle* $\triangle ABC$ formed by three noncollinear points A, B, and C.
- The *vertices*, *sides*, and *angles* of $\triangle ABC$. (The “sides” are segments, not lines.)
- The sides *opposite to* and *adjacent to* a given vertex A of $\triangle ABC$.
- Medians* of a triangle (see Figure 1.18).
- Altitudes* of a triangle (see Figure 1.19).
- Isosceles* triangle, its *base*, and its *base angles*.
- Equilateral* triangle.
- Right* triangle.

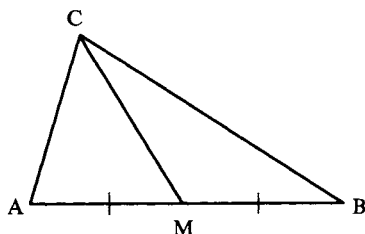
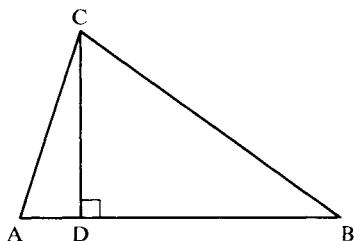
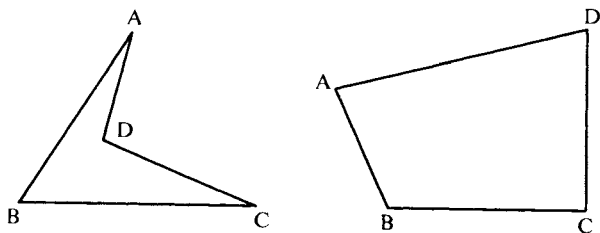


FIGURE 1.18 Median.

**FIGURE 1.19** Altitude.

3. Given four points, A, B, C, and D, no three of which are collinear and such that any pair of the segments AB, BC, CD, and DA either have no point in common or have only an endpoint in common. We can then define the *quadrilateral* $\square ABCD$ to consist of the four segments mentioned, which are called its *sides*, the four points being called its *vertices*; see Figure 1.20. (Note that the order in which the letters are written is essential. For example, $\square ABCD$ may not denote a quadrilateral, because, for example, AB might cross CD. If $\square ABCD$ did denote a quadrilateral, it would not denote the same one as $\square ACDB$. Which permutations of the four letters A, B, C, and D do denote the same quadrilateral as $\square ABCD$?) Using this definition, define the following notions:

- The *angles* of $\square ABCD$.
- Adjacent* sides of $\square ABCD$.
- Opposite* sides of $\square ABCD$.
- The *diagonals* of $\square ABCD$.
- A *parallelogram*. (Use the word “parallel.”)

**FIGURE 1.20** Quadrilaterals.

4. Define *vertical angles* (Figure 1.21). How would you attempt to prove that vertical angles are congruent to each other? (Just sketch a plan for a proof—don’t carry it out in detail.)

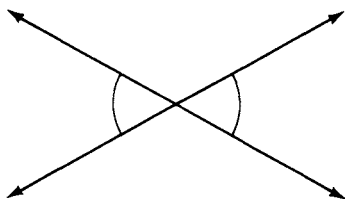


FIGURE 1.21 Vertical angles.

5. Use a common notion (p. 13) to prove the following result: If P and Q are any points on a circle with center O and radius OA , then $OP \cong OQ$.
6. (a) Given two points A and B and a third point C between them. (Recall that “between” is an *undefined* term.) Can you think of any way to prove from the postulates that C lies on line \overleftrightarrow{AB} ?
 (b) Assuming that you succeeded in proving C lies on \overleftrightarrow{AB} , can you prove from the definition of “ray” and the postulates that $\overrightarrow{AB} = \overrightarrow{AC}$?
7. If S and T are any sets, their *union* ($S \cup T$) and *intersection* ($S \cap T$) are defined as follows:
 - (i) Something belongs to $S \cup T$ if and only if it belongs either to S or to T (or to both of them).
 - (ii) Something belongs to $S \cap T$ if and only if it belongs both to S and to T .

Given two points A and B , consider the two rays \overrightarrow{AB} and \overrightarrow{BA} . Draw diagrams to show that $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$ and $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$. What additional axioms about the undefined term “between” must we assume in order to be able to *prove* these equalities?

8. To further illustrate the need for careful definition, consider the following possible definitions of *rectangle*:
 - (i) A quadrilateral with four right angles.
 - (ii) A quadrilateral with all angles congruent to one another.
 - (iii) A parallelogram with at least one right angle.

In this book we will take (i) as our definition. Your experience with Euclidean geometry may lead you to believe that these three definitions are equivalent; sketch informally how you might prove that, and notice carefully which theorems you are tacitly assuming. In hyperbolic geometry these definitions give rise to three different sets of quadrilaterals (see Chapter 6). Given the definition of “rectangle,” use it to define “square.”

9. Can you think of any way to prove from the postulates that for every line l
 - (a) There exists a point lying on l ?
 - (b) There exists a point not lying on l ?

10. Can you think of any way to prove from the postulates that the plane is nonempty, i.e., that points and lines exist? (Discuss with your instructor what it means to say that mathematical objects, such as points and lines, “exist.”)
11. Do you think that the Euclidean parallel postulate is “obvious”? Write a brief essay explaining your answer.
12. What is the flaw in the “proof” that all triangles are isosceles? (All the theorems from Euclidean geometry used in the argument are correct.)
13. If the number π is defined as the ratio of the circumference of any circle to its diameter, what theorem must first be proved to legitimize this definition? (For example, if I “define” a new number ϕ to be the ratio of the area of any circle to its diameter, that would not be legitimate. The required theorem is proved in Section 21.2 of Moise, 1990.)
14. Do you think the axiomatic method can be applied to subjects other than mathematics? Is the U.S. Constitution (including all its amendments) the list of axioms from which the federal courts logically deduce all rules of law? Do you think the “truths” asserted in the Declaration of Independence are “self-evident”?
15. Write a commentary on the application of the axiomatic method finished in 1675 by Benedict de Spinoza, entitled: *Ethics Demonstrated in Geometrical Order and Divided into Five Parts Which Treat (1) of God; (2) of the Nature and Origin of the Mind; (3) of the Nature and Origin of the Emotions; (4) of Human Bondage, or of the Strength of the Emotions; (5) of the Power of the Intellect, or of Human Liberty*. (Devote the main body of your review to Parts 4 and 5.)

MAJOR EXERCISES

1. In this exercise we will review several basic Euclidean constructions with a straightedge and compass. Such constructions fascinated mathematicians from ancient Greece until the nineteenth century, when all classical construction problems were finally solved.
 - (a) Given a segment AB. Construct the perpendicular bisector of AB. (Hint: Make AB a diagonal of a rhombus, as in Figure 1.22.)
 - (b) Given a line l and a point P lying on l . Construct the line through P perpendicular to l . (Hint: Make P the midpoint of a segment of l .)
 - (c) Given a line l and a point P *not* lying on l . Construct the line through P perpendicular to l . (Hint: Construct isosceles triangle $\triangle ABP$ with base AB on l and use (a).)

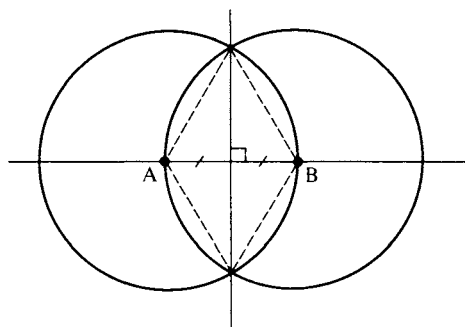


FIGURE 1.22

- (d) Given a line l and a point P not lying on l . Construct a line through P parallel to l . (Hint: use (b) and (c).)
 - (e) Construct the bisecting ray of an angle. (Hint: Use the Euclidean theorem that the perpendicular bisector of the base on an isosceles triangle is also the angle bisector of the angle opposite the base.)
 - (f) Given $\triangle ABC$ and segment $\overline{DE} \cong \overline{AB}$. Construct a point F on a given side of line \overleftrightarrow{DE} such that $\triangle DEF \cong \triangle ABC$.
 - (g) Given angle $\angle ABC$ and ray \overrightarrow{DE} . Construct F on a given side of line \overleftrightarrow{DE} such that $\angle ABC \cong \angle FDE$.
2. Euclid assumed the compass to be *collapsible*. That is, given two points P and Q , the compass can draw a circle with center P passing through Q (Postulate III); however, the spike cannot be moved to another center O to draw a circle of the same radius. Once the spike is moved, the compass collapses. Check through your constructions in Exercise 1 to see if they are possible with a collapsible compass. (For purposes of this exercise, being “given” a line means being given two or more points on it.)
- (a) Given three points P , Q , and R . Construct with a straightedge and collapsible compass a rectangle $\square PQST$ with PQ as a side and such that $PT \cong PR$ (see Figure 1.23).

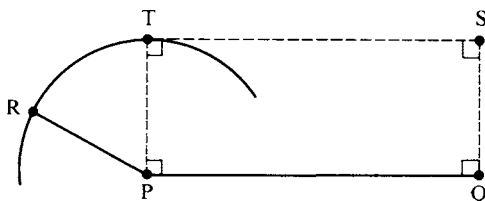


FIGURE 1.23

- (b) Given a segment PQ and a ray \overrightarrow{AB} . Construct the point C on \overrightarrow{AB} such that $PQ \cong AC$. (Hint: Using (a), construct rectangle $\square PAST$ with

$PT \cong PQ$, and then draw the circle centered at A and passing through S .)

Exercise (b) shows that you can transfer segments with a collapsible compass and a straightedge, so you can carry out all constructions as if your compass did not collapse.

3. The straightedge you used in the previous exercises was supposed to be *unruled* (if it did have marks on it, you weren't supposed to use them). Now, however, let us mark two points on the straightedge so as to mark off a certain distance d . Archimedes showed how we can then trisect an arbitrary angle:

For any angle, draw a circle γ of radius d centered at the vertex O of the angle. This circle cuts the sides of the angle at points A and B . Place the marked straightedge so that one mark gives a point C on line \overrightarrow{OA} such that O is between C and A , the other mark gives a point D on circle γ , and the straightedge must simultaneously rest on the point B , so that B , C , and D are collinear (Figure 1.24). Prove that $\angle COD$ so constructed is one-third of $\angle AOB$. (Hint: Use Euclidean theorems on exterior angles and isosceles triangles.)

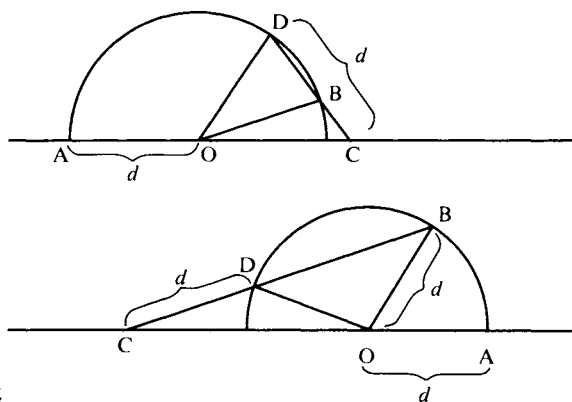


FIGURE 1.24

4. The number $\rho = (1 + \sqrt{5})/2$ was called the *golden ratio* by the Greeks, and a rectangle whose sides are in this ratio is called a *golden rectangle*.⁸ Prove that a golden rectangle can be constructed with straightedge and compass as follows:
- (a) Construct a square $\square ABCD$.

⁸ For applications of the golden ratio to Fibonacci numbers and phyllotaxis, see Coxeter (1969), Chapter 11.

- (b) Construct midpoint M of AB.
 (c) Construct point E such that B is between A and E and $MC \cong ME$ (Figure 1.25).

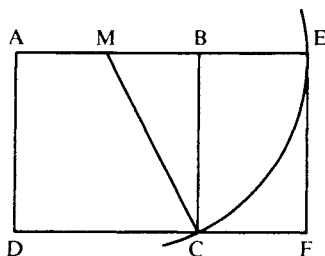


FIGURE 1.25

- (d) Construct the foot F of the perpendicular from E to \vec{DC} .
 (e) Then $\square ADFE$ is a golden rectangle (use the Pythagorean theorem for $\triangle MBC$).
 (f) Moreover, $\square BEFC$ is another golden rectangle (first show that $1/\rho = \rho - 1$).

The next two exercises require a knowledge of trigonometry.

5. The Egyptians thought that if a quadrilateral had sides of lengths a , b , c , and d , then its area S was given by the formula $(a + c)(b + d)/4$. Prove that actually

$$4S \leq (a + c)(b + d)$$

with equality holding only for rectangles. (Hint: Twice the area of a triangle is $ab \sin \theta$, where θ is the angle between the sides of lengths a , b and $\sin \theta \leq 1$, with equality holding only if θ is a right angle.)

6. Prove analogously that if a triangle has sides of lengths a , b , c , then its area S satisfies the inequality

$$4S\sqrt{3} \leq a^2 + b^2 + c^2$$

with equality holding only for equilateral triangles. (Hint: If θ is the angle between sides b and c , chosen so that it is at most 60° , then use the formulas

$$\begin{aligned} 2S &= bc \sin \theta \\ 2bc \cos \theta &= b^2 + c^2 - a^2 \quad (\text{law of cosines}) \\ \cos (60^\circ - \theta) &= (\cos \theta + \sqrt{3} \sin \theta)/2 \end{aligned}$$

7. Let $\triangle ABC$ be such that AB is not congruent to AC . Let D be the point of intersection of the bisector of $\angle A$ and the perpendicular bisector of side

BC. Let E, F, and G be the feet of the perpendiculars dropped from D to \overleftrightarrow{AB} , \overleftrightarrow{AC} , \overleftrightarrow{BC} , respectively. Prove that:

- (a) D lies outside the triangle on the circle through ABC.
- (b) One of E or F lies inside the triangle and the other outside.
- (c) E, F, and G are collinear.

(Use anything you know, including coordinates if necessary.)

PROJECTS

1. Write a paper explaining in detail why it is impossible to trisect an arbitrary angle or square a circle using only a compass and unmarked straightedge; see Jones, Morris, and Pearson (1991); Eves (1963–1965); Kutuzov (1960); or Moise (1990). Explain how arbitrary angles can be trisected if in addition we are allowed to draw a parabola or a hyperbola or a conchoid or a limaçon (see Peressini and Sherbert, 1971).
2. Here are two other famous results in the theory of constructions:
 - (a) The Danish mathematician G. Mohr and the Italian L. Mascheroni discovered independently that all Euclidean constructions of points can be made with a compass alone. A line, of course, cannot be drawn with a compass, but it can be determined with a compass by constructing two points lying on it. In this sense, Mohr and Mascheroni showed that the straightedge is unnecessary.
 - (b) On the other hand, the German J. Steiner and the Frenchman J. V. Poncelet showed that all Euclidean constructions can be carried out with a straightedge alone if we are first given a single circle and its center.

Report on these remarkable discoveries (see Eves, 1963–1965, and Kutuzov, 1960).

3. Given any $\triangle ABC$. Draw the two rays that trisect each of its angles, and let P, Q, and R be the three points of intersection of adjacent trisectors. Prove Morley's theorem⁹ that $\triangle PQR$ is an equilateral triangle (see Figure 1.26 and Coxeter, 1969).
4. An n -sided polygon is called *regular* if all its sides (respectively, angles) are congruent to one another. Construct a regular pentagon and a regular hexagon with straightedge and compass. The regular septagon cannot be so constructed; in fact, Gauss proved the remarkable theorem that the regular n -gon is constructible if and only if all odd prime factors of n occur

⁹ For a converse and generalization of Morley's theorem, see Kleven (1978).

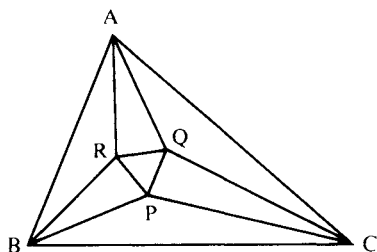


FIGURE 1.26 Morley's theorem.

to the first power and have the form $2^{2^n} + 1$ (e.g., 3, 5, 17, 257, 65,537). Report on this result, using Klein (1956). Primes of that form are called *Fermat primes*. The five listed are the only ones known at this time. Gauss did not actually construct the regular 257-gon or 65,537-gon; he only showed that the minimal polynomial equation satisfied by $\cos(2\pi/n)$ for such n could be solved in the surd field (see Moise, 1990). Other devoted (obsessive?) mathematicians carried out the constructions. The constructor for $n = 65,537$ labored for 10 years and was rewarded with a Ph.D. degree; what is the reward for checking his work?

5. Write a short biography of Archimedes (Bell, 1961, is one good reference). Archimedes discovered some of the ideas of integral calculus 14 centuries before Newton and Leibniz.