Reductio ad absurdum . . . is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

G. H. HARDY

INFORMAL LOGIC

In the previous chapter we were introduced to the postulates and basic definitions of Euclid's geometry, slightly rephrased for greater precision. We would like to begin proving some theorems or propositions that are logical consequences of the postulates. However, the exercises of the previous chapter may have alerted you to expect some difficulties that we must first clear up. For example, there is nothing in the postulates that guarantees that a line has any points lying on it (or off it)! You may feel this is ridiculous—it wouldn't be a line if it didn't have any points lying on it. (What kind of a line is he feeding us anyway?) In a sense, your protest would be legitimate, for if my concept of a line were so different from yours, we would not understand each other, and Requirement 0—that there be mutual understanding of words and symbols used—would be violated.

So let me be perfectly clear. We must play this game according to the rules, the rules mentioned in Requirement 2 but not spelled out. Unfortunately, to discuss them completely would require changing the content of this book from geometry to symbolic logic. Instead, I will only remind you of some basic rules of reasoning that you, as a rational being, already know.
LOGIC RULE 0. No unstated assumptions may be used in a proof.

The reason for taking the trouble in Chapter 1 to list all our axioms was to be explicit about our basic assumptions, including even the most obvious. Although it is "obvious" that two points determine a unique line, Euclid stated this as his first postulate. So if in some proof we want to say that every line has points lying on it, we should list this statement as another postulate (or prove it, but we can't). In other words, all our cards must be out on the table. If you reread Exercises 6, 7, 9, and 10 in Chapter 1, you will find some "obvious" assumptions that we will have to make explicit. This will be done later.

Perhaps you have realized by now that there is a vital relation between axioms and undefined terms. As we have seen, we must have undefined terms in order to avoid infinite regress. But this does not mean we can use these terms in any way we choose. The axioms tell us exactly what properties of undefined terms we are allowed to use in our arguments. You may have some other properties in your mind when you think about these terms, but you're not allowed to use them in a proof (Rule 0). For example, when you think of the unique line determined by two points, you probably think of it as being "straight," or as "the shortest path between the two points." Euclid's postulates do not allow us to assume these properties. Besides, from one viewpoint, these properties could be considered contradictory. If you were traveling the surface of the earth, say, from San Francisco to Moscow, the shortest path would be an arc of a great circle (a straight path would
bore through the earth). Indeed, pilots in a hurry fly their aircraft over great circles.

THEOREMS AND PROOFS

All mathematical theorems are conditional statements, statements of the form

\[ \text{If [hypothesis] then [conclusion].} \]

In some cases a theorem may state only a conclusion; the axioms of the particular mathematical system are then implicit (assumed) as a hypothesis. If a theorem is not written in the conditional form, it can nevertheless be translated into that form. For example,

\[ \text{Base angles of an isosceles triangle are congruent.} \]

can be interpreted as

\[ \text{If a triangle has two congruent sides, then the angles opposite those sides are congruent.} \]

Put another way, a conditional statement says that one condition (the hypothesis) implies another (the conclusion). If we denote the hypothesis by \( H \), the conclusion by \( C \), and the word “implies” by an arrow \( \Rightarrow \), then every theorem has the form \( H \Rightarrow C \). (In the example above, \( H \) is “two sides of a triangle are congruent” and \( C \) is “the angles opposite those sides are congruent.”)

Not every conditional statement is a theorem. For example, the statement

\[ \text{If } \triangle ABC \text{ is any triangle, then it is isosceles.} \]

is not a theorem. Why not? You might say that this statement is “false” whereas theorems are “true.” Let’s avoid the loaded words “true” and “false,” for they beg the question and lead us into more complicated issues.

In a given mathematical system the only statements we call theorems\(^1\) are those statements for which a proof has been supplied. We can

\(^1\) Or sometimes propositions, corollaries, or lemmas. “Theorem” and “proposition” are interchangeable; a “corollary” is an immediate consequence of a theorem, and a “lemma” is a “helping theorem.” Logically, they all mean the same; the title is just an indicator of the author’s emphasis.
disprove the assertion that every triangle is isosceles by exhibiting a triangle that is not isosceles, such as a 3-4-5 right triangle.

The crux of the matter then is the notion of proof. By definition, a proof is a list of statements, together with a justification for each statement, ending up with the conclusion desired. Usually, each statement in a proof will be numbered in this book, and the justification for it will follow in parentheses.

LOGIC RULE 1. The following are the six types of justifications allowed for statements in proofs:

1. "By hypothesis. . . ."
2. "By axiom. . . ."
4. "By definition. . . ."
5. "By step . . ." (a previous step in the argument).
6. "By rule . . . of logic."

Later in the book our proofs will be less formal, and justifications may be omitted when they are obvious (Be forewarned, however, that these omissions can lead to incorrect results.) A justification may involve several of the above types.

Having described proofs, it would be nice to be able to tell you how to find or construct them. Yet that is the mystery of doing mathematics: Certain techniques for proving theorems are learned by experience, by imitating what others have done. But there is no rote method for proving or disproving every statement in mathematics. (The non-existence of such a rote method is, when stated precisely, a deep theorem in mathematical logic and is the reason why computers will never put mathematicians out of business—see DeLong, 1970, Chapter 5).

However, some suggestions may help you construct proofs. First, make sure you clearly understand the meaning of each term in the statement of the proposed theorem. If necessary, review their definitions. Second, keep reminding yourself of what it is you are trying to prove. If it involves parallel lines, for example, look up previous propositions that give you information about parallel lines. If you find another proposition that seems to apply to the problem at hand, check carefully to see whether it really does apply. Draw pictures to help you visualize the problem.
RAA PROOFS

The most common type of proof in this book is proof by reductio ad absurdum, abbreviated RAA. In this type of proof you want to prove a conditional statement, $H \Rightarrow C$, and you begin by assuming the contrary of the conclusion you seek. We call this contrary assumption the RAA hypothesis, to distinguish it from the hypothesis $H$. The RAA hypothesis is a temporary assumption from which we derive, by reasoning, an absurd statement ("absurd" in the sense that it denies something known to be valid). Such a statement might deny the hypothesis of the theorem or the RAA hypothesis; it might deny a previously proved theorem or an axiom. Once it is shown that the negation of $C$ leads to an absurdity, it follows that $C$ must be valid. This is called the RAA conclusion. To summarize:

LOGIC RULE 2. To prove a statement $H \Rightarrow C$, assume the negation of statement $C$ (RAA hypothesis) and deduce an absurd statement, using the hypothesis $H$ if needed in your deduction.

Let us illustrate this rule by proving the following proposition (Proposition 2.1): If $l$ and $m$ are distinct lines that are not parallel, then $l$ and $m$ have a unique point in common.

Proof:
(1) Because $l$ and $m$ are not parallel, they have a point in common (by definition of "parallel").
(2) Since we want to prove uniqueness for the point in common, we will assume the contrary, that $l$ and $m$ have two distinct points $A$ and $B$ in common (RAA hypothesis).
(3) Then there is more than one line on which $A$ and $B$ both lie (step 2 and the hypothesis of the theorem, $l \neq m$).
(4) $A$ and $B$ lie on a unique line (Euclid's Postulate I).
(5) Intersection of $l$ and $m$ is unique (3 contradicts 4, RAA conclusion).

Notice that in steps 2 and 5, instead of writing "Logic Rule 2" as justification, we wrote the more suggestive "RAA hypothesis" and "RAA conclusion," respectively.
As another illustration, consider one of the earliest RAA proofs, discovered by the Pythagoreans (to their great dismay). In giving this proof, we will use some facts about Euclidean geometry and numbers that you know, and we will be informal.

Suppose \( \triangle ABC \) is a right isosceles triangle with right angle at C. We can choose our unit of length so that the legs have length 1. The theorem then says that the length of the hypotenuse is irrational (Figure 2.2).

By the Pythagorean theorem, the length of the hypotenuse is \( \sqrt{2} \), so we must prove that \( \sqrt{2} \) is an irrational number, i.e., that it is not a rational number.

What is a rational number? It is a number that can be expressed as a quotient \( \frac{p}{q} \) of two integers \( p \) and \( q \). For example, \( \frac{1}{2} \), \( \frac{2}{3} \), and \( 5 = \frac{5}{1} \) are rational numbers. We want to prove that \( \sqrt{2} \) is not one of these numbers.

We begin by assuming the contrary, that \( \sqrt{2} \) is a rational number (RAA hypothesis). In other words, \( \sqrt{2} = \frac{p}{q} \) for certain unspecified whole numbers \( p \) and \( q \). You know that every rational number can be written in lowest terms, i.e., such that the numerator and denominator have no common factor. For example, \( \frac{4}{8} \) can be written as \( \frac{2}{3} \), where the common factor 2 in the numerator and denominator has been canceled. Thus we can assume all common factors have been canceled, so that \( p \) and \( q \) have no common factor.

Next, we clear denominators:

\[
\sqrt{2} q = p
\]

and square both sides:

\[
2q^2 = p^2.
\]
This equation says that \( p^2 \) is an even number (since \( p^2 \) is twice another whole number, namely, \( q^2 \)). If \( p^2 \) is even, \( p \) must be even, for the square of an odd number is odd, as you know. Thus,

\[ p = 2r \]

for some whole number \( r \) (that is what it means to be even). Substituting \( 2r \) for \( p \) in the previous equations gives

\[ 2q^2 = (2r)^2 = 4r^2. \]

We then cancel 2 from both sides to get

\[ q^2 = 2r^2. \]

This equation says that \( q^2 \) is an even number; hence \( q \) must be even.

We have shown that numerator \( p \) and denominator \( q \) are both even, meaning that they have 2 as a common factor. Now this is absurd, because all common factors were canceled. Thus, \( \sqrt{2} \) is irrational (RAA conclusion). ■

NEGATION

In an RAA proof we begin by "assuming the contrary." Sometimes the contrary or negation of a statement is not obvious, so you should know the rules for negation.

First, some remarks on notation. If \( S \) is any statement, we will denote the negation or contrary of \( S \) by \( \sim S \). For example, if \( S \) is the statement "\( p \) is even," then \( \sim S \) is the statement "\( p \) is not even" or "\( p \) is odd."

The rule below applies to those cases where \( S \) is already a negative statement. The rule states that two negatives make a positive.

LOGIC RULE 3. The statement "\( \sim(\sim S) \)" means the same as "\( S \)."

We followed this rule when we negated the statement "\( \sqrt{2} \) is irrational" by writing the contrary as "\( \sqrt{2} \) is rational" instead of "\( \sqrt{2} \) is not irrational."
Another rule we have already followed in our RAA method is the rule for negating an implication. We wish to prove \( H \Rightarrow C \), and we assume, on the contrary, \( H \) does not imply \( C \), i.e., that \( H \) holds and at the same time \( \sim C \) holds. We write this symbolically as \( H \& \sim C \), where \& is the abbreviation for “and.” A statement involving the connective “and” is called a conjunction. Thus:

**LOGIC RULE 4.** The statement “\( \sim [H \Rightarrow C] \)” means the same as “\( H \& \sim C \).”

Let us consider, for example, the conditional statement “if 3 is an odd number, then \( 3^2 \) is even.” According to Rule 4, the negation of this is the declarative statement “3 is an odd number and \( 3^2 \) is odd.”

How do we negate a conjunction? A conjunction \( S_1 \& S_2 \) means that statements \( S_1 \) and \( S_2 \) both hold. Negating this would mean asserting that one of them does not hold, i.e., asserting the negation of one or the other. Thus:

**LOGIC RULE 5.** The statement “\( \sim [S_1 \& S_2] \)” means the same as “\( [\sim S_1 \text{ or } \sim S_2] \).”

A statement involving the connective “or” is called a disjunction. The mathematical “or” is not exclusive like “or” in everyday usage. Consider the conjunction “\( 1 = 2 \text{ and } 1 = 3 \).” If we wish to deny this, we must write (according to Rule 5) “\( 1 \neq 2 \text{ or } 1 \neq 3 \).” Of course, both inequalities are valid. So when a mathematician writes “\( S_1 \text{ or } S_2 \)” he means “either \( S_1 \) holds or \( S_2 \) holds or they both hold.”

Finally let us be more precise about what is an absurd statement. It is the conjunction of a statement \( S \) with the negation of \( S \), i.e., “\( S \& \sim S \)” A statement of this type is called a contradiction. A system of axioms from which no contradiction can be deduced is called consistent.

**QUANTIFIERS**

Most mathematical statements involve variables. For instance, the Pythagorean theorem states that for any right triangle, if \( a \) and \( b \) are the lengths of the legs and \( c \) the length of the hypotenuse, then
\[ c^2 = a^2 + b^2. \] Here \( a, b, \) and \( c \) are variable numbers, and the triangle whose sides they measure is a variable triangle.

Variables can be quantified in two different ways. First, in a universal way, as in the expressions:

- "For any \( x, \ldots .\)"
- "For every \( x, \ldots .\)"
- "For all \( x, \ldots .\)"
- "Given any \( x, \ldots .\)"
- "If \( x \) is any \( \ldots .\)"

Second, in an existential way, as in the expressions:

- "For some \( x, \ldots .\)"
- "There exists an \( x \ldots .\)"
- "There is an \( x \ldots .\)"
- "There are \( x \ldots .\)"

Consider Euclid's first postulate, which states informally that two points \( P \) and \( Q \) determine a unique line \( \ell \). Here \( P \) and \( Q \) may be any two points, so they are quantified universally, whereas \( \ell \) is quantified existentially, since it is asserted to exist, once \( P \) and \( Q \) are given.

It must be emphasized that a statement beginning with "For every \( \ldots .\)" does not imply the existence of anything. The statement "every unicorn has a horn on its head" does not imply that unicorns exist.

If a variable \( x \) is quantified universally, this is usually denoted as \( \forall x \), (read as "for all \( x \)"). If \( x \) is quantified existentially, this is usually denoted as \( \exists x \) (read as "there exists an \( x \ldots .\)"). After a variable \( x \) is quantified, some statement is made about \( x \), which we can write as \( S(x) \) (read as "statement \( S \) about \( x \)"”). Thus, a universally quantified statement about a variable \( x \) has the form \( \forall x S(x) \).

We wish to have rules for negating quantified statements. How do we deny that statement \( S(x) \) holds for all \( x \)? We can do so clearly by asserting that for some \( x \), \( S(x) \) does not hold.

LOGIC RULE 6. The statement \( \sim [\forall x S(x)] \) means the same as \( \exists x \sim S(x) \).

For example, to deny "All triangles are isosceles" is to assert "There is a triangle that is not isosceles."

Similarly, to deny that there exists an $x$ having property $S(x)$ is to assert that all $x$ fail to have property $S(x)$.

**LOGIC RULE 7.** The statement "$\neg[\exists x S(x)]$" means the same as "$\forall x \neg S(x)$."

For example, to deny "There is an equilateral right triangle" is to assert "Every right triangle is nonequilateral" or, equivalently, to assert "No right triangle is equilateral."

Since in practice quantified statements involve several variables, the above rules will have to be applied several times. Usually, common sense will quickly give you the negation. If not, follow the above rules.

Let's work out the denial of Euclid's first postulate. This postulate is a statement about all pairs of points $P$ and $Q$; negating it would mean, according to Rule 6, asserting the existence of points $P$ and $Q$ that do not satisfy the postulate. Postulate I involves a conjunction, asserting that $P$ and $Q$ lie on some line $l$ and that $l$ is unique. In order to deny this conjunction, we follow Rule 5. The assertion becomes either "$P$ and $Q$ do not lie on any line" or "they lie on more than one line." Thus, the negation of Postulate I asserts: "There are two points $P$ and $Q$ that either do not lie on any line or lie on more than one line."

If we return to the example of the surface of the earth, thinking of a "line" as a great circle, we see that there do exist such points $P$ and $Q$—namely, take $P$ to be the north pole and $Q$ the south pole. Infinitely many great circles pass through both poles. (See Figure 2.3.)

Mathematical statements are sometimes made informally, and you may sometimes have to rephrase them before you will be able to
negate them. For example, consider the following statement:

*If a line intersects one of two parallel lines, it also intersects the other.*

This appears to be a conditional statement, of the form "if . . . then . . ."; its negation, according to Rule 4, would appear to be:

*A line intersects one of two parallel lines and does not intersect the other.*

If this seems awkward, it is because the original statement contained hidden quantifiers that have been ignored. The original statement refers to *any* line that intersects one of two parallel lines, and these are *any* parallel lines. There are universal quantifiers implicit in the original statement. So we have to follow Rule 6 as well as Rule 4 in forming the correct negation, which is:

*There exist two parallel lines and a line that intersects one of them and does not intersect the other.*

**IMPLICATION**

Another rule, called the *rule of detachment*, or *modus ponens*, is the following:

**LOGIC RULE 8.** If \( P \implies Q \) and \( P \) are steps in a proof, then \( Q \) is a justifiable step.

This rule is almost a definition of what we mean by implication. For example, we have an axiom stating that if \( \measuredangle A \) and \( \measuredangle B \) are right angles, then \( \measuredangle A \equiv \measuredangle B \) (Postulate IV). Now in the course of a proof we may come across two right angles. Rule 8 allows us to assert their congruence as a step in the proof.

You should beware of confusing a conditional statement \( P \implies Q \) with its *converse* \( Q \implies P \). For example, the converse of Postulate IV states that if \( \measuredangle A \equiv \measuredangle B \) then \( \measuredangle A \) and \( \measuredangle B \) are right angles, which is absurd.

However, it may sometimes happen that both a conditional statement and its converse are valid. In case \( P \implies Q \) and \( Q \implies P \) both hold,
we write simply $P \iff Q$ (read as "$P$ if and only if $Q$" or "$P$ is logically equivalent to $Q$"). All definitions are of this form. For example, three points are collinear if and only if they lie on a line. Some theorems are also of this form, such as the theorem "a triangle is isosceles if and only if two of its angles are congruent to each other." The next rule gives a few more ways that "implication" is often used in proofs.

**LOGIC RULE 9.**

(a) $[[P \implies Q] \land [Q \implies R]] \implies [P \implies R]$.
(b) $[P \land Q] \implies P$, $[P \land Q] \implies Q$.
(c) $[\neg Q \implies \neg P] \iff [P \implies Q]$.

Part (c) states that every implication $P \implies Q$ is logically equivalent to its contrapositive $\neg Q \implies \neg P$. All parts of Rule 9 are called tautologies, because they are valid just by their form, not because of what $P$, $Q$, and $R$ mean; by contrast, the validity of a formula such as $P \implies Q$ does depend on the meaning, as we have just seen. There are infinitely many tautologies, and the next rule gives the most infamous.

**LAW OF EXCLUDED MIDDLE AND PROOF BY CASES**

**LOGIC RULE 10.** For every statement $P$, "$P$ or $\neg P$" is a valid step in a proof (law of excluded middle).²

For example, given point $P$ and line $l$, we may assert that either $P$ lies on $l$ or it does not. If this is a step in a proof, we will usually then break the rest of the proof into cases — giving an argument under the case assumption that $P$ lies on $l$ and giving another argument under the case assumption that $P$ does not. Both arguments must be given, or

²The law of excluded middle characterizes classical two-valued logic; either a statement holds or it does not; there is no middle ground such as "we don't know." Constructivist mathematicians (such as Brouwer, Bishop, Beeson, and Stolzenberg) reject the unqualified use of this rule when applied to existence statements. They insist that in order to meaningfully prove that a mathematical object exists, one must supply an effective method for constructing it. It is uninformative merely to assume that the object does not exist (RAA hypothesis) and then derive a contradiction (so they also reject Logic Rule 6 when applied to infinite sets). The "constructive" aspect of Euclid's geometry traditionally refers to "straightedge and compass constructions" (see the Major Exercises of Chapter 1). We will pay close attention to this aspect throughout this book.
else the proof is incomplete. A proof of this type is given for Proposition 3.16 in Chapter 3, which asserts that there exists a line through \( P \) perpendicular to \( I \).

Sometimes there are more than two cases. For example, it is a theorem that either an angle is acute or it is right or it is obtuse — three cases. We will have to give three arguments — one for each case assumption. You will give such arguments when you prove the SSS criterion for congruence of triangles in Exercise 32 of Chapter 3. This method of \textit{proof by cases} was used (correctly) in the incorrect attempt in Chapter 1 to prove that all triangles are isosceles.

LOGIC RULE 11. Suppose the disjunction of statements \( S_1 \) or \( S_2 \) or \ldots or \( S_n \) is already a valid step in a proof. Suppose that proofs of \( C \) are carried out from each of the \textit{case assumptions} \( S_1, S_2, \ldots, S_n \). Then \( C \) can be concluded as a valid step in the proof (proof by cases).

And this concludes our discussion of logic. No claim is made that all the rules of logic have been listed, just that those listed should suffice for our purposes. For further discussion, see DeLong (1970) and his bibliography.

**INCIDENCE GEOMETRY**

Let us apply the logic we have developed to a very elementary part of geometry, \textit{incidence geometry}. We assume only the undefined terms "point" and "line" and the undefined relation "incidence" between a point and a line, written "\( P \) lies on \( I \)" or \( P \in I \) or "\( I \) passes through \( P \)" as before. We don't discuss "betweenness" or "congruence" in this restricted geometry (but we are now beginning the new axiomatic development of geometry that fills the gaps in Euclid and applies to other geometries as well; that development will continue in future chapters, and the formal definitions given in Chapter 1 will be used).

These undefined terms will be subjected to three axioms, the first of which is the same as Euclid's first postulate.

**INCIDENCE AXIOM 1.** For every point \( P \) and for every point \( Q \) not equal to \( P \) there exists a unique line \( I \) incident with \( P \) and \( Q \).
INCIDENCE AXIOM 2. For every line \( I \) there exist at least two distinct points incident with \( I \).

INCIDENCE AXIOM 3. There exist three distinct points with the property that no line is incident with all three of them.

These axioms fill the gap mentioned in Exercises 9 and 10, Chapter 1. We can now assert that every line has points lying on it — at least two, possibly more — and that the points do not all lie on one line. Moreover, we know that the geometry must have at least three points in it, by the third axiom and Rule 9(b) of logic. Namely, Incidence Axiom 3 is a conjunction of two statements:

1. There exist distinct points \( A, B, \) and \( C \).
2. For every line, at least one of these points does not lie on the line.

Rule 9(b) tells us that a conjunction of two statements implies each statement separately, so we can conclude that three distinct points exist (Rule 8).

Incidence geometry has some defined terms, such as “collinear,” “concurrent,” and “parallel,” defined exactly as they were in Chapter 1. Incidence Axiom 3 can be rewritten as “there exist three noncollinear points.” Parallel lines are still lines that do not have a point in common.

What sort of results can we prove using this meager collection of axioms? None that are very exciting, but here are a few you can prove as exercises.

PROPOSITION 2.1. If \( I \) and \( m \) are distinct lines that are not parallel, then \( I \) and \( m \) have a unique point in common.

PROPOSITION 2.2. There exist three distinct lines that are not concurrent.

PROPOSITION 2.3. For every line there is at least one point not lying on it.

PROPOSITION 2.4. For every point there is at least one line not passing through it.

PROPOSITION 2.5. For every point \( P \) there exist at least two lines through \( P \).
MODELS

In reading over the axioms of incidence in the previous section, you may have imagined dots and long dashes drawn on a sheet of paper. With this representation in mind, the axioms appear to be correct statements. We will take the point of view that these dots and dashes are a model for incidence geometry.

More generally, if we have any axiom system, we can interpret the undefined terms in some way, i.e., give the undefined terms a particular meaning. We call this an interpretation of the system. We can then ask whether the axioms, so interpreted, are correct statements. If they are, we call the interpretation a model. When we take this point of view, interpretations of the undefined terms “point,” “line,” and “incident” other than the usual dot-and-dash drawings become possible.

Example 1. Consider a set \{A, B, C\} of three letters, which we will call “points.” “Lines” will be those subsets that contain exactly two letters — \{A, B\}, \{A, C\}, and \{B, C\}. A “point” will be interpreted as “incident” with a “line” if it is a member of that subset. Thus, under this interpretation, A lies on \{A, B\} and \{A, C\} but does not lie on \{B, C\}. In order to determine whether this interpretation is a model, we must check whether the interpretations of the axioms are correct statements. For Incidence Axiom 1, if P and Q are any two of the letters, A, B, and C, \{P, Q\} is the unique “line” on which they both lie. For Axiom 2, if \{P, Q\} is any “line,” P and Q are two distinct “points” lying on it. For Axiom 3, we see that A, B, and C are three distinct “points” that are not collinear.

What is the use of models? The main property of any model of an axiom system is that all theorems of the system are correct statements in the model. This is because logical consequences of correct statements are themselves correct. (By definition of “model,” axioms are correct statements when interpreted in models; theorems are logical consequences of axioms.) Thus, we immediately know that the five propositions in the previous section hold in the three-point geometry above (Example 1).
Suppose we have a statement in the formal system but don’t yet know whether it is a theorem, i.e., we don’t yet know whether it can be proved. We can look at our models and see whether the statement is correct in the models. If we can find one model where the interpreted statement fails to hold, we can be sure that no proof is possible. You are undoubtedly familiar with testing for the correctness of geometric statements by drawing pictures. Of course, the converse does not work; just because a drawing makes a statement look right does not mean you can prove it. This was illustrated on pp. 23–25.

The advantage of having several models is that a statement may hold in one model but not in another. Models are “laboratories” for experimenting with the formal system.

Let us experiment with the Euclidean parallel postulate. This is a statement in the formal system incidence geometry: “For every line \( I \) and every point \( P \) not lying on \( I \) there exists a unique line through \( P \) that is parallel to \( I \).” This statement appears to be correct according to our drawings (although we cannot verify the uniqueness of the parallelism, since we cannot extend our dashes indefinitely). But what about our three-point model? It is immediately apparent that no parallel lines exist in this model: \( \{A, B\} \) meets \( \{B, C\} \) in the point \( B \) and meets \( \{A, C\} \) in the point \( A \); \( \{B, C\} \) meets \( \{A, C\} \) in the point \( C \). (We say that this model has the elliptic parallel property.)

Thus, we can conclude that no proof of the Euclidean parallel postulate from the axioms of incidence alone is possible; in fact, in incidence geometry it is impossible to prove that parallel lines exist. Similarly, the statement “any two lines have a point in common” (the elliptic parallel property) cannot be proved from the axioms of incidence geometry, for if you

\[ \text{FIGURE 2.4 Elliptic parallel property (no parallel lines). A 3-point incidence geometry.} \]
could prove it, it would hold in the usual drawn model (and in the models that will be described in Examples 3 and 4).

The technical description for this situation is that the statement "parallel lines exist" is "independent" of the axioms of incidence. We call a statement independent of given axioms if it is impossible to either prove or disprove the statement from the axioms. Independence is demonstrated by constructing two models for the axioms: one in which the statement holds and one in which it does not hold. This method will be used very decisively in Chapter 7 to settle once and for all the question of whether the parallel postulate can be proved.

An axiom system is called complete if there are no independent statements in the language of the system, i.e., every statement in the language of the system can either be proved or disproved from the axioms. Thus, the axioms for incidence geometry are incomplete. The axioms for Euclidean and hyperbolic geometries given later in the book can be proved to be complete (see Tarski's article in Henkin, Suppes, and Tarski, 1959).

**Example 2.** Suppose we interpret "points" as points on a sphere, "lines" as great circles on the sphere, and "incidence" in the usual sense, as a point lying on a great circle. In this interpretation there are again no parallel lines. However, this interpretation is not a model for incidence geometry, for, as was already mentioned, the interpretation of Incidence Axiom 1 fails to hold — there are an infinite number of great circles passing through the north and south poles on the sphere (see Figure 2.3).

**Example 3.** Let the "points" be the four letters A, B, C, and D. Let the "lines" be all six sets containing exactly two of these letters:

![Figure 2.5](image)

Euclidean parallel property. A 4-point incidence geometry.
FIGURE 2.8 Hyperbolic parallel property. A 5-point incidence geometry.

\{(A, B), (A, C), (A, D), (B, C), (B, D), and (C, D)\}. Let "incidence" be set membership, as in Example 1. As an exercise, you can verify that this is a model for incidence geometry and that in this model the Euclidean parallel postulate does hold (see Figure 2.5).

**Example 4.** Let the "points" be the five letters A, B, C, D, and E. Let the "lines" be all 10 sets containing exactly two of these letters. Let "incidence" be set membership, as in Examples 1 and 3. You can verify that in this model the following statement about parallel lines, characteristic of hyperbolic geometry, holds: "For every line \( l \) and every point \( P \) not on \( l \) there exist at least two lines through \( P \) parallel to \( l \)." (See Figure 2.6).

Let us summarize the significance of models. Models can be used to prove the independence of a statement from given axioms; i.e., models can be used to demonstrate the impossibility of proving or disproving a statement from the axioms. Moreover, if an axiom system has many models that are essentially different from each other, as the models in Examples 1, 3, and 4 are essentially different from each other, then that system has a wide range of applicability. Propositions proved from the axioms of such a system are automatically correct statements within any of the models. Mathematicians often discover

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3 An incidence geometry with only finitely many points is called a *finite geometry*. There is an entertaining discussion of finite geometries (with applications to growing tomato plants) in Chapter 4 of Beck, Bleicher, and Crowe (1969). For an advanced treatment, see Dembowski (1968) or Stevenson (1972). See the exercises at the end of this chapter for more examples.
that an axiom system constructed with one particular model in mind has applications to completely different models never dreamed of.

At the other extreme, when all models of an axiom system are isomorphic to one another, the axioms are called categorical. (The axioms for Euclidean and hyperbolic geometries given later in the book are categorical.) The advantage of categorical axioms is that they completely describe all properties of the model that are expressible in the language of the system. 4 (For a simple example of a categorical system, suppose we add to the three incidence axioms a fourth axiom asserting that there do not exist four distinct points. Obviously, the three-point model in Example 1 is the only model, up to isomorphism, for this expanded axiom system.)

Finally, models provide evidence for the consistency of the axiom system. For example, if incidence geometry were inconsistent, the supposed proof of a contradiction could be translated into proof of a contradiction in the utterly trivial set theory for the set of three letters A, B, and C (Example 1).

**ISOMORPHISM OF MODELS**

We want to make precise the notion of two models being "essentially the same" or isomorphic: for incidence geometries, this will mean that there exists a one-to-one correspondence $P \leftrightarrow P'$ between the points of the models and a one-to-one correspondence $l \leftrightarrow l'$ between the lines of the models such that $P$ lies on $l$ if and only if $P'$ lies on $l'$; such a correspondence is called an *isomorphism* from one model onto the other.

**Example 5.** Consider a set $(a, b, c)$ of three letters, which we will call "lines" now. "Points" will be those subsets that contain exactly two letters — $(a, b), (a, c),$ and $(b, c)$. Let incidence be set membership; for example, "point" $(a, b)$ is incident with "line" $a$ and "line" $b$, not

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4 This is a nontrivial (and nonconstructive) theorem of mathematical logic called Gödel’s completeness theorem, which says (modulo cardinality considerations) that if the system is categorical, then for every sentence $S$, there exists either a proof of $S$ or a proof of $\neg S$. 

with “line” \( c \). This model certainly seems to be structurally the same as the three-point model in Example 1—all we’ve changed is the notation. An explicit isomorphism is given by the following correspondences:

\[
\begin{align*}
\text{A} & \leftrightarrow \{a, b\} & \text{A, B} & \leftrightarrow b \\
\text{B} & \leftrightarrow \{b, c\} & \text{B, C} & \leftrightarrow c \\
\text{C} & \leftrightarrow \{a, c\} & \text{A, C} & \leftrightarrow a
\end{align*}
\]

Note that A lies on \( \{A, B\} \) and \( \{A, C\} \) only; its corresponding “point” \( \{a, b\} \) lies on the corresponding “lines” \( b \) and \( a \) only. Similar checking with B and C shows that incidence is preserved by our correspondence. On the other hand, if we used a correspondence such as

\[
\begin{align*}
\{A, B\} & \leftrightarrow a \\
\{B, C\} & \leftrightarrow b \\
\{A, C\} & \leftrightarrow c
\end{align*}
\]

for the “lines,” keeping the same correspondence for the “points,” we would not have an isomorphism because, for example, A lies on \( \{A, C\} \) but the corresponding “point” \( \{a, b\} \) does not lie on the corresponding “line” \( c \).

To further illustrate the idea that isomorphic models are “essentially the same,” consider two models with different parallelism properties, such as one with the elliptic property and one with the Euclidean. We claim that these models are not isomorphic: suppose, on the contrary, that an isomorphism could be set up. Given line \( l \) and point \( P \) not on it; then every line through \( P \) meets \( l \), by the elliptic property. Hence every line through the corresponding point \( P' \) meets the corresponding line \( l' \), but that contradicts the Euclidean property of the second model.

Later on, we will need to use the concept of “isomorphism” for models of a geometry more complicated than incidence geometry—neutral geometry. In neutral geometry we will have betweenness and congruence relations, in addition to the incidence relation, and we will require an “isomorphism” to preserve those relations as well.

The general idea is that an isomorphism of two models of an axiom system is a one-to-one correspondence between the basic objects of the system that preserves all the basic relations of the system.

Another example to be discussed in Chapter 9 is the axiom system for a “group.” Roughly speaking, a group is a set with a multiplication
for its elements satisfying a few familiar axioms of algebra. An "isomorphism" of groups will then be a one-to-one mapping $x \rightarrow x'$ of one set onto the other which preserves the multiplication, i.e., for which $(xy)' = x'y'$.

**PROJECTIVE AND AFFINE PLANES**

We now very briefly discuss two types of models of incidence geometry that are particularly significant. During the Renaissance, around the fifteenth century, artists developed a theory of perspective in order to realistically paint two-dimensional representations of three-dimensional scenes. The theory described the projection of points in the scene onto the artist's canvas by lines from those points to a fixed viewing point in the artist's eye; the intersection of those lines with the plane of the canvas was used to construct the painting. The mathematical formulation of this theory was called *projective geometry*.

In this technique of projection, parallel lines that lie in a plane cutting the plane of the canvas are painted as meeting (visually, they appear to meet at a point on the horizon). This suggested an extension of Euclidean geometry in which parallel lines "meet at infinity," so that the Euclidean parallel property is replaced by the elliptic parallel property in the extended plane. We will carry out this extension rigorously. First, some definitions.

**DEFINITION.** A *projective plane* is a model of the incidence axioms having the elliptic parallel property (any two lines meet) and such that every line has at least three distinct points lying on it (strengthened Incidence Axiom 2).

Our proposed extension of the Euclidean plane uses only its incidence properties (not its betweenness and congruence properties); the purely incidence part of Euclidean geometry is called *affine geometry*, which leads to the next definition.

**DEFINITION.** An *affine plane* is a model of incidence geometry having the Euclidean parallel property.
Example 3 in this chapter illustrated the smallest affine plane (four points, six lines).

Let $\mathcal{A}$ be any affine plane. We introduce a relation $l \sim m$ on the lines of $\mathcal{A}$ to mean "$l = m$ or $l \parallel m$." This relation is obviously reflexive ($l \sim l$) and symmetric ($l \sim m \Rightarrow m \sim l$). Let us prove that it is transitive ($l \sim m$ and $m \sim n \Rightarrow l \sim n$): if any pair of these lines are equal, the conclusion is immediate, so assume that we have three distinct lines such that $l \parallel m$ and $m \parallel n$. Suppose, on the contrary, that $l$ meets $n$ at point $P$. $P$ does not lie on $m$, because $l \parallel m$. Hence we have two distinct parallels $n$ and $l$ to $m$ through $P$, which contradicts the Euclidean parallel property of $\mathcal{A}$.

A relation which is reflexive, symmetric, and transitive is called an equivalence relation. Such relations occur frequently in mathematics and are very important. Whenever they occur, we consider the equivalence classes determined by the relation: for example, the equivalence class $[l]$ of $l$ is defined to be the set consisting of all lines equivalent to $l$—i.e., of $l$ and all the lines in $\mathcal{A}$ parallel to $l$. In the familiar Cartesian model of the Euclidean plane, the set of all horizontal lines is one equivalence class, the set of verticals is another, the set of lines with slope 1 is a third, and so on. Equivalence classes take us from equivalence to equality: $l \sim m \iff [l] = [m]$.

For historical and visual reasons, we call these equivalence classes points at infinity; we have made this vague idea precise within modern set theory. We now enlarge the model $\mathcal{A}$ to a new model $\mathcal{A}^*$ by adding these points, calling the points of $\mathcal{A}$ "ordinary" points for emphasis. We further enlarge the incidence relation by specifying that each of these equivalence classes lies on every one of the lines in that class: $[l]$ lies on $l$ and on every line $m$ such that $l \parallel m$. Thus, in the enlarged plane $\mathcal{A}^*$, $l$ and $m$ are no longer parallel, but they meet at $[l]$.

We want $\mathcal{A}^*$ to be a model of incidence geometry also, which requires one more step. To satisfy Euclid's Postulate I, we need to add one new line on which all (and only) the points at infinity lie: define the line at infinity $l_\infty$ to be the set of all points at infinity. Let us now check that $\mathcal{A}^*$ is a projective plane, called the projective completion of $\mathcal{A}$:

**Verification of I-1.** If $P$ and $Q$ are ordinary points, they lie on a unique line of $\mathcal{A}$ (since I-1 holds in $\mathcal{A}$) and they do not lie on $l_\infty$. If $P$ is
ordinary and \( Q \) is a point at infinity \([m]\), then either \( P \) lies on \( m \) and \( \overrightarrow{PQ} = m \), or, by the Euclidean parallel property, \( P \) lies on a unique parallel \( n \) to \( m \) and \( Q \) also lies on \( n \) (by definition of incidence for points at infinity), so \( \overrightarrow{PQ} = n \). If both \( P \) and \( Q \) are points at infinity, then \( \overrightarrow{PQ} = l_\infty \).

**Verification of Strengthened I-2.** Each line \( m \) of \( A \) has at least two points on it (by I-2 in \( A \)), and now we’ve added a third point \([m]\) at infinity. That \( l_\infty \) has at least three points on it follows from the existence in \( A \) of three lines that intersect in pairs (such as the lines joining the three noncollinear points furnished by Axiom I-3); the equivalence classes of those three lines do the job.

**Verification of I-3.** It holds already in \( A \).

**Verification of the Elliptic Parallel Property.** If two ordinary lines do not meet in \( A \), then they belong to the same equivalence class and meet at that point at infinity. An ordinary line \( m \) meets \( l_\infty \) at \([m]\).

**Example 6.** Figure 2.7 illustrates the smallest projective plane, projective completion of the smallest affine plane; it has seven points and seven lines. The dashed line could represent the line at infinity, for removing it and the three points \( C \), \( B \), and \( E \) that lie on it leaves us with a four-point, 6-line affine plane isomorphic to the one in Example 3, Figure 2.5.

\[
\begin{array}{c}
\text{FIGURE 2.7} \quad \text{The smallest projective plane (7 points).}
\end{array}
\]
The usual Euclidean plane, regarded just as a model of incidence geometry (ignoring its betweenness and congruence structures), is called the real affine plane, and its projective completion is called the real projective plane. Coordinate descriptions of these planes are given in Major Exercises 9 and 10; other models isomorphic to the real projective plane are described in Exercise 10(c), and a “curved” model isomorphic to the real affine plane is described in Major Exercise 5.

**Example 7.** To visualize the projective completion \( \mathbb{A}^* \) of the real affine plane \( \mathbb{A} \), picture \( \mathbb{A} \) as the plane \( T \) tangent to a sphere \( S \) in Euclidean three-space at its north pole \( N \) (Figure 2.8). If \( O \) is the center of sphere \( S \), we can join each point \( P \) of \( T \) to \( O \) by a Euclidean line that will intersect the northern hemisphere of \( S \) in a unique point \( P' \); this gives a one-to-one correspondence between the points \( P \) of \( T \) and the points \( P' \) of the northern hemisphere of \( S \) (\( N \) corresponds to itself). Similarly, given any line \( m \) of \( T \), we join \( m \) to \( O \) by a plane \( \Pi \) through \( O \) that cuts out a great circle on the sphere and a great
semicircle \( m' \) on the northern hemisphere; this gives a one-to-one correspondence between the lines \( m \) of \( T \) and the great semicircles \( m' \) of the northern hemisphere, a correspondence that clearly preserves incidence.

Now if \( l \parallel m \) in \( T \), the planes through \( O \) determined by these parallel lines will meet in a line lying in the plane of the equator, a line which (since it goes through \( O \)) cuts out a pair of antipodal points on the equator. Thus the line at infinity of \( \mathcal{A}^* \) can be visualized under our isomorphism as the equator of \( S \) with antipodal points identified (they must be identified, or else Axiom I-1 will fail). In other words, \( \mathcal{A}^* \) can be described as the northern hemisphere with antipodal points on the equator pasted to each other; however, we can’t visualize this pasting very well, because it can be proved that the pasting cannot be done in Euclidean three-space without tearing the hemisphere.

Projective planes are the most important models of pure incidence geometry. We will see later on that Euclidean, hyperbolic, and, of course, elliptic geometry can all be considered “subgeometries” of projective geometry. This discovery by Cayley led him to exclaim that “projective geometry is all of geometry,” which turned out to be an oversimplification.

**REVIEW EXERCISE**

Which of the following statements are correct?

1. The “hypothesis” of a theorem is an assumption that implies the conclusion.
2. A theorem may be proved by drawing an accurate diagram.
3. To say that a step is “obvious” is an allowable justification in a rigorous proof.
4. There is no way to program a computer to prove or disprove every statement in mathematics.
5. To “disprove” a statement means to prove the negation of that statement.
6. A “model” of an axiom system is the same as an “interpretation” of the system.
7. The Pythagoreans discovered the existence of irrational lengths by an RAA proof.
(8) The negation of the statement "If 3 is an odd number, then 9 is even" is the statement "If 3 is an odd number, then 9 is odd."

(9) The negation of a conjunction is a disjunction.

(10) The statement "1 = 2 and 1 ≠ 2" is an example of a contradiction.

(11) The statement "Base angles of an isosceles triangle are congruent" has no hidden quantifiers.

(12) The statements "Some triangles are equilateral" and "There exists an equilateral triangle" have the same meaning.

(13) The converse of the statement "If you push me, then I will fall" is the statement "If you push me, then I won't fall."

(14) The following two statements are logically equivalent: If \( l \parallel m \), then \( l \) and \( m \) have no point in common. If \( l \) and \( m \) have a point in common, then \( l \) and \( m \) are not parallel.

(15) Whenever a conditional statement is valid, its converse is also valid.

(16) If one statement implies a second statement and the second statement implies a third statement, then the first statement implies the third statement.

(17) The negation of "All triangles are isosceles" is "No triangles are isosceles."

(18) The hyperbolic parallel property is defined as "For every line \( l \) and every point \( P \) not on \( l \) there exist at least two lines through \( P \) parallel to \( l \)."

(19) The statement "Every point has at least two lines passing through it" is independent of the axioms for incidence geometry.

(20) "If \( l \parallel m \) and \( m \parallel n \), then \( l \parallel n \)" is independent of the axioms of incidence geometry.

**EXERCISES**

1. Let \( S \) be the following self-referential statement: "Statement \( S \) is false." Show that if \( S \) is either true or false then there is a contradiction in our language. (This is the liar paradox. Kurt Gödel used a variant of it as the starting point for his famous incompleteness theorem in logic; see Delong, 1970)

2. (a) What is the negation of \([P \lor Q]\)?
   (b) What is the negation of \([P \land \neg Q]\)?
   (c) Using the rules of logic given in the text, show that \( P \Rightarrow Q \) means the same as \([\neg P \lor Q]\). (Hint: Show they are both negations of the same thing.)
(d) A symbolic way of writing Rule 2 for RAA proofs is 
\[ [[H \land \neg C] \Rightarrow [S \land \neg S]] \Rightarrow [H \Rightarrow C]. \]
Explain this.

3. Negate Euclid’s fourth postulate.

5. Write out the converse to the following statements:
   (a) “If lines \(l\) and \(m\) are parallel, then a transversal \(t\) to lines \(l\) and \(m\) cuts out congruent alternate interior angles.”
   (b) “If the sum of the degree measures of the interior angles on one side of transversal \(t\) is less than 180°, then lines \(l\) and \(m\) meet on that side of transversal \(t\).”

6. Prove all five propositions in incidence geometry as stated in this chapter. Don’t use Incidence Axiom 2 in your proofs.

7. For each pair of axioms of incidence geometry, construct an interpretation in which those two axioms are satisfied but the third axiom is not. (This will show that the three axioms are independent, in the sense that it is impossible to prove any one of them from the other two.)

8. Show that the interpretations in Examples 3 and 4 in this chapter are models of incidence geometry and that the Euclidean and hyperbolic parallel properties, respectively, hold.

9. In each of the following interpretations of the undefined terms, which of the axioms of incidence geometry are satisfied and which are not? Tell whether each interpretation has the elliptic, Euclidean, or hyperbolic parallel property.
   (a) “Points” are dots on a sheet of paper, “lines” are circles drawn on the paper, “incidence” means that the dot lies on the circle.
   (b) “Points” are lines in Euclidean three-dimensional space, “lines” are planes in Euclidean three-space, “incidence” is the usual relation of a line lying in a plane.
   (c) Same as in (b), except that we restrict ourselves to lines and planes that pass through a fixed ordinary point \(O\).
   (d) Fix a circle in the Euclidean plane. Interpret “point” to mean an ordinary Euclidean point inside the circle, interpret “line” to mean a chord of the circle, and let “incidence” mean that the point lies on the chord in the usual sense. (A chord of a circle is a segment whose endpoints lie on the circle.)
   (e) Fix a sphere in Euclidean three-space. Two points on the sphere are called antipodal if they lie on a diameter of the sphere; e.g., the north and south poles are antipodal. Interpret a “point” to be a set \(\{P, P'\}\) consisting of two antipodal points on the sphere. Interpret a “line” to be a great circle \(C\) on the sphere. Interpret a “point” \(\{P, P'\}\) to “lie on” a “line” \(C\) if one of the points \(P, P'\) lies on the great circle \(C\) (then the other point also lies on \(C\)).
10. (a) Prove that when each of two models of incidence geometry has exactly three "points" in it, the models are isomorphic.
   (b) Must two models having exactly four "points" be isomorphic? If you think so, prove this; if you think not, give a counterexample.
   (c) Show that the models in Exercises 9(c) and 9(e) are isomorphic. (Hint: Take the point O of Exercise 9(c) to be the center of the sphere in Exercise 9(e), and cut the sphere with lines and planes through point O to get the isomorphism.)

11. Construct a model of incidence geometry that has neither the elliptic, hyperbolic, nor Euclidean parallel properties. (These properties refer to any line / and any point P not on /. Construct a model that has different parallelism properties for different choices of / and P. Five points suffice.)

12. Suppose that in a given model for incidence geometry every "line" has at least three distinct "points" lying on it. What are the least number of "points" and the least number of "lines" such a model can have? Suppose further that the model has the Euclidean parallel property. Show that 9 is now the least number of "points" and 12 the least number of "lines" such a model can have.

13. The following syllogisms are by Lewis Carroll. Which of them are correct arguments?
   (a) No frogs are poetical; some ducks are unpoetical. Hence, some ducks are not frogs.
   (b) Gold is heavy; nothing but gold will silence him. Hence, nothing light will silence him.
   (c) All lions are fierce; some lions do not drink coffee. Hence, some creatures that drink coffee are not fierce.
   (d) Some pillows are soft; no pokers are soft. Hence, some pokers are not pillows.

14. Comment on the following example of isomorphic structures given by a music student: *Romeo and Juliet* and *West Side Story*.

15. Comment on the following statement by the artist David Hunter: "The only use for Logic is writing books on Logic and teaching courses in Logic; it has no application to human behavior."

**MAJOR EXERCISES**

1. Let $M$ be a projective plane. Define a new interpretation $M'$ by taking as "points" of $M'$ the lines of $M$ and as "lines" of $M'$ the points of $M$, with the same incidence relation. Prove that $M'$ is also a projective plane.
(called the dual plane of \( M \)). Suppose further that \( M \) has only finitely many points. Prove that all the lines in \( M \) have the same number of points lying on them. (Hint: See Figure 7.43 in Chapter 7.)

2. Let us add to the axioms of incidence geometry the following axioms:
   (i) The Euclidean parallel property.
   (ii) The existence of only a finite number of points.
   (iii) The existence of lines \( l \) and \( m \) such that the number of points lying on \( l \) is different from the number of points lying on \( m \).

Show that this expanded axiom system is inconsistent. (Hint: Prove that (i) and (ii) imply the negation of (iii).)

3. Prove that every projective plane \( \mathcal{P} \) is isomorphic to the projective completion of some affine plane \( \mathcal{A} \). (Hint: As was done in Example 6, pick any line \( m \) in \( \mathcal{B} \), pretend that \( m \) is "the line at infinity," remove \( m \) and the points lying on it, and prove that what's left is an affine plane \( \mathcal{A} \) and that \( \mathcal{B} \) is isomorphic to the completion \( \mathcal{A}^* \).) A surprising discovery is that \( \mathcal{A} \) need not be unique up to isomorphism (see Hartshorne, 1967).

4. Provide another solution to Major Exercise 2 by embedding the affine plane of that exercise in its completion and invoking Major Exercise 1.

5. Consider the following interpretation of incidence geometry. Begin with a punctured sphere in Euclidean three-space, i.e., a sphere with one point \( N \) removed. Interpret "points" as points on the punctured sphere. For each circle on the original sphere passing through \( N \), interpret the punctured circle obtained by removing \( N \) as a "line." Interpret "incidence" in the Euclidean sense of a point lying on a punctured circle. Is this interpretation a model? If so, what parallel property does it have? Is it isomorphic to any other model you know? (Hint: If \( N \) is the north pole, project the punctured sphere from \( N \) onto the plane \( \Pi \) tangent to the sphere at the south pole, as in Figure 2.9. Use the fact that planes through \( N \) cut out circles on the sphere and lines in \( \Pi \). For a hilarious discussion of this interpretation, refer to Chapter 3 of Sved, 1991.)
6. Consider the following statement in incidence geometry: "For any two lines \( I \) and \( m \) there exists a one-to-one correspondence between the set of points lying on \( I \) and the set of points lying on \( m \)." Prove that this statement is independent of the axioms of incidence geometry.

7. Let \( M \) be a finite projective plane so that, according to Major Exercise 1, all lines in \( M \) have the same number of points lying on them; call this number \( n + 1 \). Prove the following:
   (a) Each point in \( M \) has \( n + 1 \) lines passing through it.
   (b) The total number of points in \( M \) is \( n^2 + n + 1 \).
   (c) The total number of lines in \( M \) is \( n^2 + n + 1 \).

8. Let \( A \) be a finite affine plane so that, according to Major Exercise 2, all lines in \( A \) have the same number of points lying on them; call this number \( n \). Prove the following:
   (a) Each point in \( A \) has \( n + 1 \) lines passing through it.
   (b) The total number of points in \( A \) is \( n^2 \).
   (c) The total number of lines in \( A \) is \( n(n + 1) \).
   (Hint: Use Major Exercise 7.)

9. The real affine plane has as its "points" all ordered pairs \( (x, y) \) of real numbers. A "line" is determined by an ordered triple \((u, v, w)\) of real numbers such that either \( u \neq 0 \) or \( v \neq 0 \), and it is defined as the set of all "points" \((x, y)\) satisfying the linear equation \( ux + vy + wz = 0 \). "Incidence" is defined as set membership. Verify that all axioms for an affine plane are satisfied by this interpretation.

10. A "point" \([x, y, z]\) in the real projective plane is determined by an ordered triple \((x, y, z)\) of real numbers that are not all zero, and it consists of all the ordered triples of the form \((kx, ky, kz)\) for all real numbers \( k \neq 0 \); thus, \([kx, ky, kz] = [x, y, z]\). A "line" in the real projective plane is determined by an ordered triple \((u, v, w)\) of real numbers that are not all zero, and it is defined as the set of all "points" \([x, y, z]\) whose coordinates satisfy the linear equation \( ux + vy + wz = 0 \). "Incidence" is defined as set membership. Verify that all the axioms for a projective plane are satisfied by this interpretation. Prove that by taking \( z = 0 \) as the equation of the "line at infinity," by assigning the affine "point" \((x, y)\) the "homogeneous coordinates" \([x, y, 1]\), and by assigning affine "lines" to projective "lines" in the obvious way, the real projective plane becomes isomorphic to the projective completion of the real affine plane. Prove that the models in Exercise 10(c) are also isomorphic to the real projective plane.

11. (a) Given an interpretation of some axioms, in order to show that the interpretation is a model, you must verify that the interpretations of the axioms hold. If you execute that verification precisely rather
than casually, you are actually giving proofs. In what axiomatic theory are those proofs given? Consider this question more specifically for the models presented in the text and exercises of this chapter.

(b) Some of the interpretations refer to a “sphere” in “Euclidean space,” presuming that you already know the theory of such things, yet we are carefully laying the axiomatic foundations of the simpler theory of the Euclidean plane. Does this bother you? Comment.

(c) Can an inconsistent system (such as the one in Major Exercise 2) have a model? Explain.

12. Just because every step in a proof has been justified, that doesn’t guarantee the correctness of the proof: the justifications may be in error. For example, the justification may not be one of the six types allowed by Logic Rule 1, or it may refer to a previous theorem that is not applicable, or it may draw erroneous inferences from a definition (such as “parallel lines are equidistant”). Thus a second “proof” should be given to verify the correctness of the justifications in the first proof. But then how can we be certain the second “proof” is correct? Do we have to give a third “proof” and so on ad infinitum? Discuss.

PROJECTS

1. The following statement is by the French mathematician G. Desargues: “If the vertices of two triangles correspond in such a way that the lines joining corresponding vertices are concurrent, then the intersections of corresponding sides are collinear.” (See Figure 2.10.) This statement is independent of the axioms for projective planes: it holds in the real projective plane, but there exist other projective planes in which it fails. Report on this independence result (see Artzy, 1965, or Stevenson, 1972).

2. An isomorphism of a projective plane $\mathcal{M}$ onto its dual plane $\mathcal{M}'$ (see Major Exercise 1) is called a polarity of $\mathcal{M}$. By definition of “isomorphism,” it assigns to each point $A$ of $\mathcal{M}$ a line $p(A)$ of $\mathcal{M}$ called the polar of $A$, and to each line $m$ of $\mathcal{M}$ a point $P(m)$ of $\mathcal{M}$ called its pole, in such a way that $A$ lies on $m$ if and only if $P(m)$ lies on $p(A)$. The conic $\gamma$ determined by this polarity is defined to be the set of all points $A$ such that $A$ lies on its polar $p(A)$; $p(A)$ is defined to be the tangent line to the conic at $A$. Point $B$ is defined to be interior to $\gamma$ if every line through $B$ intersects $\gamma$ in two
points. This very abstract definition of "conic" can be reconciled with more familiar descriptions, such as (using coordinates) the solution set to a homogeneous quadratic equation in three variables. The theory of conics is one of the most important topics in plane projective geometry. Report on this, using some good projective geometry text such as Coxeter (1960). A polarity will play a crucial role in Chapter 7 (see also Major Exercise 13, Chapter 6).

3. Aristotle is considered the founder of classical logic. Up through the 1930s, some important logicians were Leibniz, Boole, Frege, Russell, Whitehead, Hilbert, Ackermann, Skolem, Gödel, Church, Tarski, and Kleene. Report on some of the history of logic, using DeLong (1970) and his bibliography as references.

The poet Goethe said: "Mathematicians are like Frenchmen: whatever you say to them, they translate it into their own language and forthwith it is something entirely different."