CHAPTER 3

HILBERT'S AXIOMS

The value of Euclid's work as a masterpiece of logic has been very grossly exaggerated.

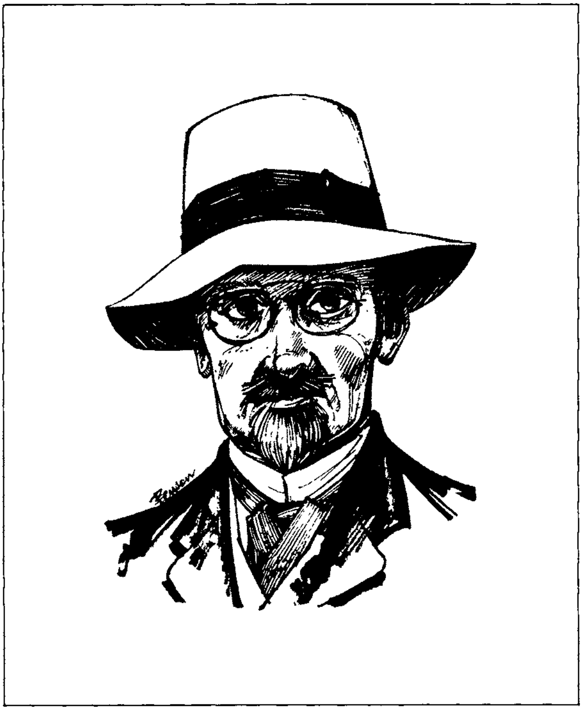
BERTRAND RUSSELL

FLAWS IN EUCLID

Having clarified our rules of reasoning (Chapter 2), let us return to the postulates of Euclid. In Exercises 9 and 10 of Chapter 1 we saw that Euclid neglected to state his assumptions that points and lines exist, that not all points are collinear, and that every line has at least two points lying on it. We made these assumptions explicit in Chapter 2 by adding two more axioms of incidence to Euclid's first postulate.

In Exercises 6 and 7, Chapter 1, we saw that some assumptions about "betweenness" are needed. In fact, Euclid never mentioned this notion explicitly, but tacitly assumed certain facts about it that are obvious in diagrams. In Chapter 1 we saw the danger of reasoning from diagrams, so these tacit assumptions will have to be made explicit.

Quite a few of Euclid's proofs are based on reasoning from diagrams. To make these proofs rigorous, a much larger system of explicit axioms is needed. Many such axiom systems have been proposed. We will present a modified version of David Hilbert's system of axioms.



David Hilbert

Hilbert's system was not the first, but his axioms are perhaps the most intuitive and are certainly the closest in spirit to Euclid's.[1]

During the first quarter of the twentieth century Hilbert was considered the leading mathematician of the world.[2] He made outstanding, original contributions to a wide range of mathematical fields as well as to physics. He is perhaps best known for his research in the foundations of geometry as well as the foundations of algebraic number theory, infinite-dimensional spaces, and mathematical logic. A

[1] Let us not forget that no serious work toward constructing new axioms for Euclidean geometry had been done until the discovery of non-Euclidean geometry shocked mathematicians into reexamining the foundations of the former. We have the paradox of non-Euclidean geometry helping us to better understand Euclidean geometry!

[2] I heartily recommend the warm and colorful biography of Hilbert by Constance Reid (1970). It is nontechnical and conveys the excitement of the time when Göttingen was the capital of the mathematical world.

great champion of the axiomatic method, he "axiomatized" all of the above subjects except for physics (although he did succeed in providing physicists with very valuable mathematical techniques). He was also a mathematical prophet; in 1900 he predicted 23 of the most important mathematical problems of this century.

He has been quoted as saying: "One must be able to say at all times—instead of points, lines and planes—tables, chairs and beer mugs." In other words, since no properties of points, lines, and planes may be used in a proof other than the properties given by the axioms, you may as well call these undefined entities by other names.

Hilbert's axioms are divided into five groups: incidence, betweenness, congruence, continuity, and parallelism. We have already seen the three axioms of incidence in Chapter 2. In the next sections we will deal successively with the other groups of axioms.

AXIOMS OF BETWEENNESS

To further illustrate the need for axioms of betweenness, consider the following attempted proof of the theorem that base angles of an isosceles triangle are congruent. This is not Euclid's proof, which is flawed in other ways (see Golos, 1968, p. 57), but is an argument found in some high school geometry texts.

Proof:
Given $\triangle ABC$ with $AC \cong BC$. To prove $\sphericalangle A \cong \sphericalangle B$ (see Figure 3.1):

(1) Let the bisector of $\sphericalangle C$ meet AB at D (every angle has a bisector).

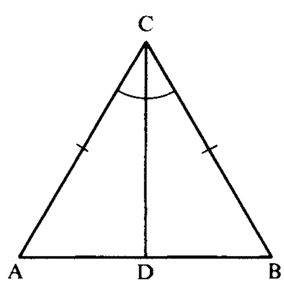


FIGURE 3.1

(2) In triangles $\triangle ACD$ and $\triangle BCD$, $AC \cong BC$ (hypothesis).
(3) $\sphericalangle ACD \cong \sphericalangle BCD$ (definition of bisector of an angle).
(4) $CD \cong CD$ (things that are equal are congruent).
(5) $\triangle ACD \cong \triangle BCD$ (SAS). (3, 3, 3)
(6) Therefore, $\sphericalangle A \cong \sphericalangle B$ (corresponding angles of congruent triangles). ■

Consider the first step, whose justification is that every angle has a bisector. This is a correct statement and can be proved separately. But how do we know that the bisector of $\sphericalangle C$ meets $\overleftrightarrow{AB}$, or if it does, how do we know that the point of intersection D lies *between* A and B? This may seem obvious, but if we are to be rigorous, it requires proof. For all we know, the picture might look like Figure 3.2. If this were the case, steps 2–5 would still be correct, but we could conclude only that $\sphericalangle B$ is congruent to $\sphericalangle CAD$, not to $\sphericalangle CAB$, since $\sphericalangle CAD$ is the angle in $\triangle ACD$ that corresponds to $\sphericalangle B$.

Once we state our four axioms of betweenness, it will be possible to prove (after a considerable amount of work) that the bisector of $\sphericalangle C$ does meet $\overleftrightarrow{AB}$ in a point D between A and B, so the above argument will be repaired (see the crossbar theorem, later in this section). There is, however, an easier proof of the theorem (given in the next section). We will use the shorthand notation

$$A * B * C$$

to abbreviate the statement "point B is between point A and point C."

BETWEENNESS AXIOM 1. If $A * B * C$, then A, B, and C are three distinct points all lying on the same line, and $C * B * A$.

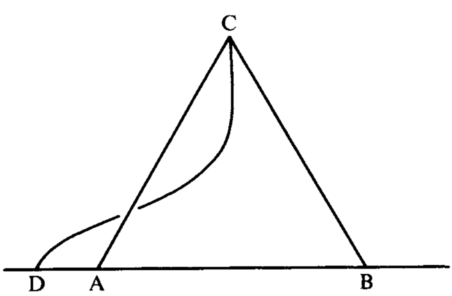


FIGURE 3.2

FIGURE 3.3

The first part of this axiom fills the gap mentioned in Exercise 6, Chapter 1. The second part (C * B * A) makes the obvious remark that "between A and C" means the same as "between C and A"—it doesn't matter whether A or C is mentioned first.

BETWEENNESS AXIOM 2. Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that A * B * D, B * C * D, and B * D * E (Figure 3.3).

This axiom ensures that there are points between B and D and that the line $\overleftrightarrow{BD}$ does not end at either B or D.

BETWEENNESS AXIOM 3. If A, B, and C are three distinct points lying on the same line, then one and only one of the points is between the other two.

This axiom ensures that a line is not circular; if the points were on a circle, you would then have to say that each is between the other two (or none is between the other two—it would depend on which of the two arcs you look at—see Figure 3.4).

Before stating the last betweenness axiom, let us examine some consequences of the first three. Recall that the *segment* AB is defined as the set of all points between A and B together with the endpoints A and B. The *ray* $\overrightarrow{AB}$ is defined as the set of all points on the segment AB together with all points C such that A * B * C. The second axiom ensures that such points as C exist, so the ray $\overrightarrow{AB}$ is larger than the segment AB. We can now prove the formulas you encountered in Exercise 7, Chapter 1.

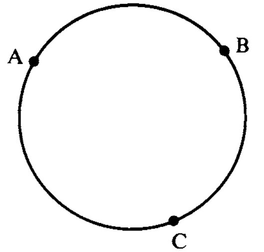


FIGURE 3.4

PROPOSITION 3.1. For any two points A and B: (i) $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$, and (ii) $\overrightarrow{AB} \cup \overrightarrow{BA} = \{\overleftrightarrow{AB}\}$.

Proof of (i):
(1) By definition of segment and ray, $AB \subset \overrightarrow{AB}$ and $AB \subset \overrightarrow{BA}$, so by definition of intersection, $AB \subset \overrightarrow{AB} \cap \overrightarrow{BA}$.
(2) Conversely, let the point C belong to the intersection of $\overrightarrow{AB}$ and $\overrightarrow{BA}$; we wish to show that C belongs to AB.
(3) If C = A or C = B, C is an endpoint of AB. Otherwise, A, B, and C are three collinear points (by definition of ray and Axiom 1), so exactly one of the relations $A * C * B$, $A * B * C$, or $C * A * B$ holds (Axiom 3).
(4) If $A * B * C$ holds, then C is not on $\overrightarrow{BA}$; if $C * A * B$ holds, then C is not on $\overrightarrow{AB}$. In either case, C does not belong to both rays.
(5) Hence, the relation $A * C * B$ must hold, so C belongs to AB. ■

The proof of (ii) is similar and is left as an exercise. (Recall that $\{\overleftrightarrow{AB}\}$ is the set of points lying on the line $\overleftrightarrow{AB}$.)

Recall next that if $C * A * B$, then $\overrightarrow{AC}$ is said to be *opposite* to $\overrightarrow{AB}$ (see Figure 3.5). By Axiom 1, points A, B, and C are collinear, and by Axiom 3, C does not belong to $\overrightarrow{AB}$, so rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are distinct. This definition is therefore in agreement with the definition given in Chapter 1 (see Proposition 3.6). Axiom 2 guarantees that every ray $\overrightarrow{AB}$ has an opposite ray $\overrightarrow{AC}$.

It seems clear from Figure 3.5 that every point P lying on the line l through A, B, C must belong either to ray $\overrightarrow{AB}$ or to an opposite ray $\overrightarrow{AC}$. This statement seems similar to the second assertion of Proposition 3.1, but it is actually more complicated; we are now discussing *four* points A, B, C, and P, whereas previously we had to deal with only three points at a time. In fact, we encounter here another "pictorially obvious" assertion that cannot be proved without introducing another axiom (see Exercise 17).

FIGURE 3.5

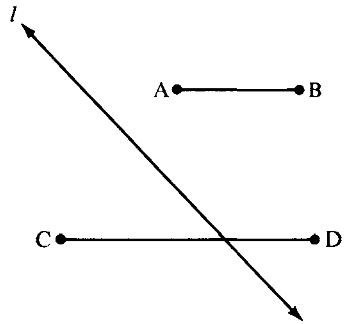


FIGURE 3.6 A and B are on the same side of l; C and D are on opposite sides of l.

Suppose we call the assertion "$C * A * B$ and P collinear with A, B, C $\Rightarrow P \in \overrightarrow{AC} \cup \overrightarrow{AB}$" *the line separation property.* Some mathematicians take this property as another axiom. However, it is considered inelegant in mathematics to assume more axioms than are necessary (although we pay for elegance by having to work harder to prove results that appear obvious). So we will not assume the line separation property as an axiom; instead, we will prove it as a consequence of our previous axioms and our last betweenness axiom, called *the plane separation axiom.*

DEFINITION. Let l be any line, A and B any points that do not lie on l. If $A = B$ or if segment AB contains no point lying on l, we say A and B are *on the same side of l,* whereas if $A \neq B$ and segment AB does intersect l, we say that A and B are *on opposite sides* of l (see Figure 3.6). The law of the excluded middle (Rule 10) tells us that A and B are either on the same side or on opposite sides of l.

BETWEENNESS AXIOM 4 (Plane Separation). For every line l and for any three points A, B, and C not lying on l:

(i) If A and B are on the same side of l and B and C are on the same side of l, then A and C are on the same side of l (see Figure 3.7).

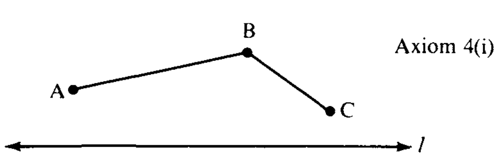


FIGURE 3.7

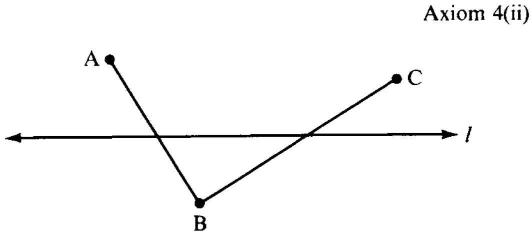


FIGURE 3.8

(ii) If A and B are on opposite sides of l and B and C are on opposite sides of l, then A and C are on the same side of l (see Figure 3.8).

COROLLARY. (iii) If A and B are on opposite sides of l and B and C are on the same side of l, then A and C are on opposite sides of l.

Axiom 4(i) indirectly guarantees that our geometry is two-dimensional, since it does not hold in three-space. (Line l could be outside the plane of this page and cut through segment AC; this interpretation shows that if we assumed the line separation property as an axiom, we could not prove the plane separation property.) Betweenness Axiom 4 is also needed to make sense out of Euclid's fifth postulate, which talks about two lines meeting on one "side" of a transversal. We can now define a *side* of a line l as the set of all points that are on the same side of l as some particular point A not lying on l. If we denote this side by H_A, notice that if C is on the same side of l as A, then by Axiom 4(i), $H_C = H_A$. (The definition of a *side* may seem circular because we use the word "side" twice, but it is not; we have already defined the compound expression "on the same side.") Another expression commonly used for a "side of l" is a *half-plane bounded by l.*

PROPOSITION 3.2. Every line bounds exactly two half-planes and these half-planes have no point in common.

Proof:

(1) There is a point A not lying on l (Proposition 2.3).
(2) There is a point O lying on l (Incidence Axiom 2).
(3) There is a point B such that B * O * A (Betweenness Axiom 2).
(4) Then A and B are on opposite sides of l (by definition), so l has at least two sides.

(5) Let C be any point distinct from A and B and not lying on l. If C and B are not on the same side of l, then C and A are on the same side of l (by the law of excluded middle and Betweenness Axiom 4(ii)). So the set of points not on l is the union of the side H_A of A and the side H_B of B.
(6) If C were on both sides (RAA hypothesis), then A and B would be on the same side (Axiom 4(i)), contradicting step 4; hence the two sides are disjoint (RAA conclusion). ■

We next apply the plane separation property to study betweenness relations among four points.

PROPOSITION 3.3. Given A * B * C and A * C * D. Then B * C * D and A * B * D. (See Figure 3.9.)

Proof:
(1) A, B, C, and D are four distinct collinear points (see Exercise 1).
(2) There exists a point E not on the line through A, B, C, D (Proposition 2.3).
(3) Consider line $\overleftrightarrow{EC}$. Since (by hypothesis) AD meets this line in point C, A and D are on opposite sides of $\overleftrightarrow{EC}$.
(4) We claim A and B are on the same side of $\overleftrightarrow{EC}$. Assume on the contrary that A and B are on opposite sides of $\overleftrightarrow{EC}$ (RAA hypothesis).
(5) Then $\overleftrightarrow{EC}$ meets $\overleftrightarrow{AB}$ in a point between A and B (definition of "opposite sides").
(6) That point must be C (Proposition 2.1).
(7) Thus, A * B * C and A * C * B, which contradicts Betweenness Axiom 3.

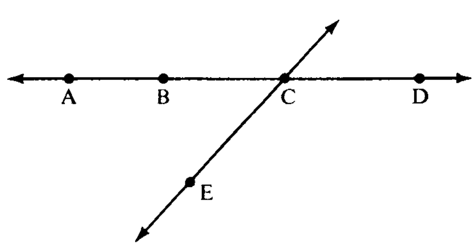


FIGURE 3.9

(8) Hence, A and B are on the same side of $\overleftrightarrow{EC}$ (RAA conclusion).
(9) B and D are on opposite sides of $\overleftrightarrow{EC}$ (steps 3 and 8 and the corollary to Betweenness Axiom 4).
(10) Hence, the point C of intersection of lines $\overleftrightarrow{EC}$ and $\overleftrightarrow{BD}$ lies between B and D (definition of "opposite sides"; Proposition 2.1, i.e., that the point of intersection is unique).

A similar argument involving $\overleftrightarrow{EB}$ proves that A * B * D (Exercise 2(b)). ■

COROLLARY. Given A * B * C and B * C * D. Then A * B * D and A * C * D.

Finally we prove the *line separation property*.

PROPOSITION 3.4. If C * A * B and *l* is the line through A, B, and C (Betweenness Axiom 1), then for every point P lying on *l*, P lies either on ray $\overrightarrow{AB}$ or on the opposite ray $\overrightarrow{AC}$.

Proof:
(1) Either P lies on $\overrightarrow{AB}$ or it does not (law of excluded middle).
(2) If P does lie on $\overrightarrow{AB}$, we are done, so assume it doesn't; then P * A * B (Betweenness Axiom 3).
(3) If P = C then P lies on $\overrightarrow{AC}$ (by definition), so assume P $\neq$ C; then exactly one of the relations C * A * P, C * P * A, and P * C * A holds (Betweenness Axiom 3 again).
(4) Suppose the relation C * A * P holds (RAA hypothesis).
(5) We know (by Betweenness Axiom 3) that exactly one of the relations P * C * B, C * P * B, and C * B * P holds.
(6) If P * B * C, then combining this with P * A * B (step 2) gives A * B * C (Proposition 3.3), contradicting the hypothesis.
(7) If C * P * B, then combining this with C * A * P (step 4) gives A * P * B (Proposition 3.3), contradicting step 2.
(8) If B * C * P, then combining this with B * A * C (hypothesis and Betweenness Axiom 1) gives A * C * P (Proposition 3.3), contradicting step 4.
(9) Since we obtain a contradiction in all three cases, C * A * P does not hold (RAA conclusion).
(10) Therefore, C * P * A or P * C * A (step 3), which means that P lies on the opposite ray $\overrightarrow{AC}$. ■

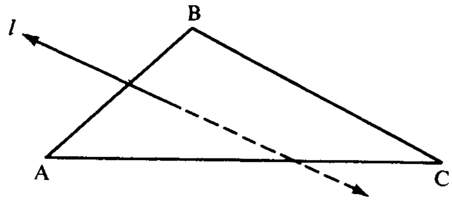


FIGURE 3.10

The next theorem states a visually obvious property that Pasch discovered Euclid to be using without proof.

PASCH'S THEOREM. If A, B, C are distinct noncollinear points and l is any line intersecting AB in a point between A and B, then l also intersects either AC or BC (see Figure 3.10). If C does not lie on l, then l does not intersect both AC and BC.

Intuitively, this theorem says that if a line "goes into" a triangle through one side, it must "come out" through another side.

Proof:

(1) Either C lies on l or it does not; if it does, the theorem holds (law of excluded middle).
(2) A and B do not lie on l, and the segment AB does intersect l (hypothesis and Axiom 1).
(3) Hence, A and B lie on opposite sides of l (by definition).
(4) From step 1 we may assume that C does not lie on l, in which case C is either on the same side of l as A or on the same side of l as B (separation axiom).
(5) If C is on the same side of l as A, then C is on the opposite side from B, which means that l intersects BC and does not intersect AC; similarly if C is on the same side of l as B, then l intersects AC and does not intersect BC (separation axiom).
(6) The conclusions of Pasch's theorem hold (Logic Rule 11—proof by cases). ■

Here are some more results on betweenness and separation that you will be asked to prove in the exercises.

PROPOSITION 3.5. Given $A * B * C$. Then $AC = AB \cup BC$ and B is the only point common to segments AB and BC.

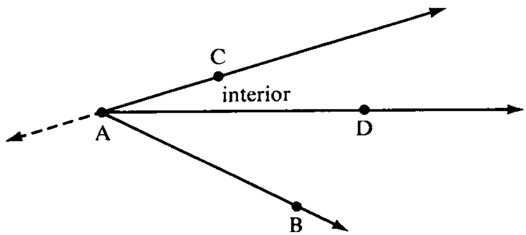


FIGURE 3.11

PROPOSITION 3.6. Given A * B * C. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.

DEFINITION. Given an angle ∢CAB, define a point D to be in the *interior* of ∢CAB if D is on the same side of $\overleftrightarrow{AC}$ as B and if D is also on the same side of $\overleftrightarrow{AB}$ as C. (Thus, the interior of an angle is the intersection of two half-planes.) See Figure 3.11.

PROPOSITION 3.7. Given an angle ∢CAB and point D lying on line $\overleftrightarrow{BC}$. Then D is in the interior of ∢CAB if and only if B * D * C (see Figure 3.12).

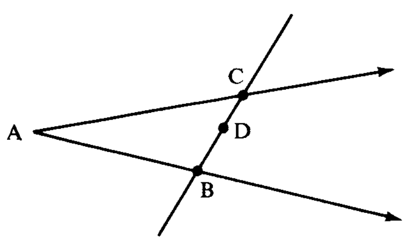


FIGURE 3.12

Warning. Do not assume that every point in the interior of an angle lies on a segment joining a point on one side of the angle to a point on the other side. In fact, this assumption is false in hyperbolic geometry (see Exercise 36).

PROPOSITION 3.8. If D is in the interior of ∢CAB; then: (a) so is every other point on ray $\overrightarrow{AD}$ except A; (b) no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of ∢CAB; and (c) if C * A * E, then B is in the interior of ∢DAE (see Figure 3.13).

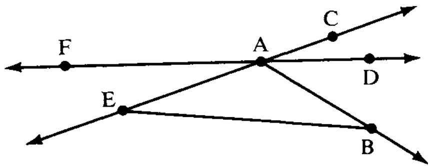


FIGURE 3.13

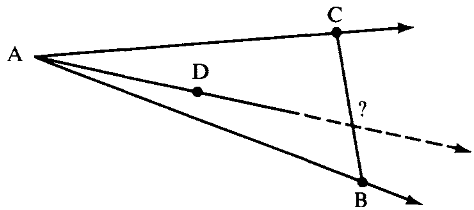


FIGURE 3.14

DEFINITION. Ray $\overrightarrow{AD}$ is *between* rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is interior to $\sphericalangle$CAB. (By Proposition 3.8(a), this definition does not depend on the choice of point D on $\overrightarrow{AD}$.)

CROSSBAR THEOREM. If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC (see Figure 3.14).

DEFINITION. The *interior* of a triangle is the intersection of the interiors of its three angles. Define a point to be *exterior* to the triangle if it is not in the interior and does not lie on any side of the triangle.

PROPOSITION 3.9. (a) If a ray *r* emanating from an exterior point of $\triangle$ABC intersects side AB in a point between A and B, then *r* also intersects side AC or side BC. (b) If a ray emanates from an interior point of $\triangle$ABC, then it intersects one of the sides, and if it does not pass through a vertex, it intersects only one side.

AXIOMS OF CONGRUENCE

If we were more pedantic, "congruent," the last of our undefined terms, would be replaced by two terms, since it refers to either a relation between segments or a relation between angles. We are accustomed to congruence as a relation between triangles, but we can now define this as follows: two triangles are *congruent* if a one-to-one correspondence can be set up between their vertices so that corresponding sides are congruent and corresponding angles are congruent. When we write $\triangle ABC \cong \triangle DEF$ we understand that A corresponds to D, B to E, and C to F. Similar definitions can be given for congruence of quadrilaterals, pentagons, and so forth.

CONGRUENCE AXIOM 1. If A and B are distinct points and if A′ is any point, then for each ray *r* emanating from A′ there is a *unique* point B′ on *r* such that $B' \neq A'$ and $AB \cong A'B'$. (See Figure 3.15.)

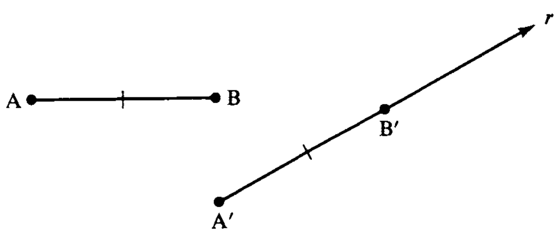


FIGURE 3.15

Intuitively speaking, this axiom says you can "move" the segment AB so that it lies on the ray *r* with A superimposed on A′, and B superimposed on B′. (In Major Exercise 2, Chapter 1, you showed how to do this with a straightedge and collapsible compass.)

CONGRUENCE AXIOM 2. If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

This axiom replaces Euclid's first common notion, since it says that segments congruent to the same segment are congruent to each other. It also replaces the fourth common notion, since it says that segments that coincide are congruent.

CONGRUENCE AXIOM 3. If $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$. (See Figure 3.16.)

This axiom replaces the second common notion, since it says that if congruent segments are "added" to congruent segments, the sums are congruent. Here, "adding" means juxtaposing segments along the same line. For example, using Congruence Axioms 1 and 3, you can

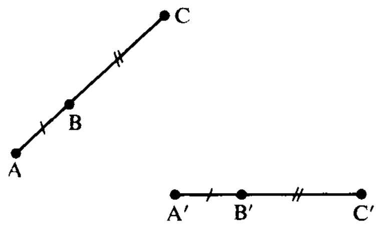


FIGURE 3.16

lay off a copy of a given segment AB two, three, . . . , *n* times, to get a new segment *n* · AB. (See Figure 3.17.)

CONGRUENCE AXIOM 4. Given any $\sphericalangle$BAC (where, by definition of "angle," $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$), and given any ray $\overrightarrow{A'B'}$ emanating from a point A′, then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\sphericalangle B'A'C' \cong \sphericalangle BAC$. (See Figure 3.18.)

This axiom can be paraphrased to state that a given angle can be "laid off" on a given side of a given ray in a unique way (see Major Exercise 1(g), Chapter 1).

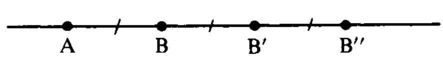


FIGURE 3.17 AB″ = 3 · AB.

CONGRUENCE AXIOM 5. If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

This is the analogue for angles of Congruence Axiom 2 for segments; the first part asserts the transitivity and the second part the reflexivity of the congruence relation. Combining them, we can prove the symmetry of this relation: $\sphericalangle A \cong \sphericalangle B \Rightarrow \sphericalangle B \cong \sphericalangle A$.

Proof:
$\sphericalangle A \cong \sphericalangle B$ (hypothesis) and $\sphericalangle A \cong \sphericalangle A$ (reflexivity) imply (substituting A for C in Congruence Axiom 5) $\sphericalangle B \cong \sphericalangle A$ (transitivity). ■

(By the same argument, congruence of segments is a symmetric relation.)

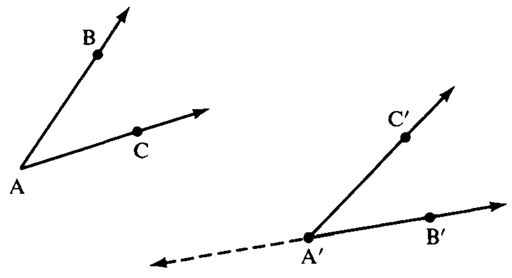


FIGURE 3.18

It would seem natural to assume next an "addition axiom" for congruence of *angles* analogous to Congruence Axiom 3 (the addition axiom for congruence of segments). We won't do this, however, because such a result can be proved using the next congruence axiom (see Proposition 3.19).

CONGRUENCE AXIOM 6 (SAS). If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent (see Figure 3.19).

This side-angle-side criterion for congruence of triangles is a profound axiom. It provides the "glue" which binds the relation of congruence of segments to the relation of congruence of angles. It enables us to deduce all the basic results about triangle congruence with which you are presumably familiar. For example, here is one immediate consequence which states that we can "lay off" a given triangle on a given base and a given half-plane.

COROLLARY TO SAS. Given $\triangle ABC$ and segment $DE \cong AB$, there is a unique point F on a given side of line $\overleftrightarrow{DE}$ such that $\triangle ABC \cong \triangle DEF$.

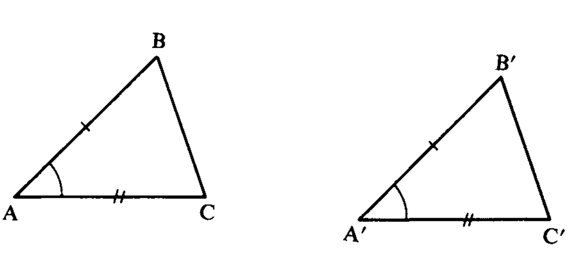


FIGURE 3.19

Proof:
There is a unique ray $\overrightarrow{DF}$ on the given side such that $\sphericalangle CAB \cong \sphericalangle FDE$, and F on that ray can be chosen to be the unique point such that $AC \cong DF$ (by Congruence Axioms 4 and 1). Then $\triangle ABC \cong \triangle DEF$ (SAS). ■

As we said, Euclid did not take SAS as an axiom but tried to prove it as a theorem. His argument was essentially as follows. Move $\triangle A'B'C'$ so as to place point A′ on point A and $\overrightarrow{A'B'}$ on $\overrightarrow{AB}$. Since $AB \cong A'B'$, by hypothesis, point B′ must fall on point B. Since $\sphericalangle A \cong \sphericalangle A'$, $\overrightarrow{A'C'}$ must fall on $\overrightarrow{AC}$, and since $AC \cong A'C'$, point C′ must coincide with point C. Hence, B′C′ will coincide with BC and the remaining angles will coincide with the remaining angles, so the triangles will be congruent.

This argument is called *superposition*. It derives from the experience of drawing two triangles on paper, cutting out one, and placing it on top of the other. Although this is a good way to convince a novice in geometry to accept SAS, it is not a proof, and Euclid reluctantly used it in only one other theorem. It is not a proof because Euclid never stated an axiom that allows figures to be moved around without changing their size and shape.

Some modern writers introduce "motion" as an undefined term and lay down axioms for this term. (In fact, in Pieri's foundations of geometry, "point" and "motion" are the only undefined terms.) Or else, the geometry is first built up on a different basis, "distances" introduced, and a "motion" defined as a one-to-one transformation of the plane onto itself that preserves distance. Euclid can be vindicated by either approach. In fact, Felix Klein, in his 1872 *Erlanger Programme,* defined a geometry as the study of those properties of figures that remain invariant under a particular group of transformations. This idea will be developed in Chapter 9.

You will show in Exercise 35 that it is impossible to prove SAS or any of the other criteria for congruence of triangles (SSS, ASA, SAA) from the preceding axioms. As usual, the method for proving the impossibility of proving some statement S is to invent a model for the preceding axioms in which S is false.

As an application of SAS, the simple proof of Pappus (A.D.300) for the theorem on base angles of an isosceles triangle follows.

PROPOSITION 3.10. If in $\triangle ABC$ we have $AB \cong AC$, then $\sphericalangle B \cong \sphericalangle C$ (see Figure 3.20).

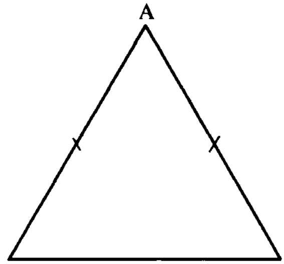


FIGURE 3.20

Proof:

(1) Consider the correspondence of vertices $A \leftrightarrow A$, $B \leftrightarrow C$, $C \leftrightarrow B$. Under this correspondence, two sides and the included angle of $\triangle ABC$ are congruent respectively to the corresponding sides and included angle of $\triangle ACB$ (by hypothesis and Congruence Axiom 5 that an angle is congruent to itself).

(2) Hence, $\triangle ABC \cong \triangle ACB$ (SAS), so the corresponding angles, $\sphericalangle B$ and $\sphericalangle C$, are congruent (by definition of congruence of triangles). ■

Here are some more familiar results on congruence. We will prove some of them; if the proof is omitted, see the exercises.

PROPOSITION 3.11 (Segment Subtraction). If $A * B * C$, $D * E * F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$ (see Figure 3.21).

PROPOSITION 3.12. Given $AC \cong DF$, then for any point B between A and C, there is a unique point E between D and F such that $AB \cong DE$.

Proof:

(1) There is a unique point E on $\overrightarrow{DF}$ such that $AB \cong DE$ (Congruence Axiom 1).

(2) Suppose E were not between D and F (RAA hypothesis; see Figure 3.22).

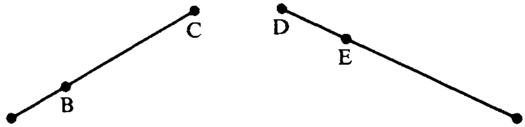


FIGURE 3.21

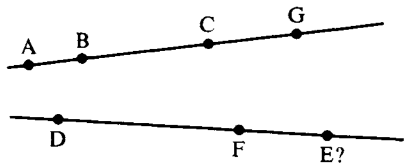


FIGURE 3.22

(3) Then either E = F or D * F * E (definition of $\overrightarrow{DF}$).
(4) If E = F, then B and C are two distinct points on $\overrightarrow{AC}$ such that AC ≅ DF ≅ AB (hypothesis, step 1), contradicting the uniqueness part of Congruence Axiom 1.
(5) If D * F * E, then there is a point G on the ray opposite to $\overrightarrow{CA}$ such that FE ≅ CG (Congruence Axiom 1).
(6) Then AG ≅ DE (Congruence Axiom 3).
(7) Thus, there are two distinct points B and G on $\overrightarrow{AC}$ such that AG ≅ DE ≅ AB (steps 1, 5, and 6), contradicting the uniqueness part of Congruence Axiom 1.
(8) D * E * F (RAA conclusion). ■

DEFINITION. AB < CD (or CD > AB) means that there exists a point E between C and D such that AB ≅ CE.

PROPOSITION 3.13 (Segment Ordering). (a) Exactly one of the following conditions holds (*trichotomy*): AB < CD, AB ≅ CD, or AB > CD. (b) If AB < CD and CD ≅ EF, then AB < EF. (c) If AB > CD and CD ≅ EF, then AB > EF. (d) if AB < CD and CD < EF, then AB < EF (transitivity).

PROPOSITION 3.14. Supplements of congruent angles are congruent.

PROPOSITION 3.15. (a) Vertical angles are congruent to each other. (b) An angle congruent to a right angle is a right angle.

PROPOSITION 3.16. For every line *l* and every point P there exists a line through P perpendicular to *l*.

Proof:
(1) Assume first that P does not lie on *l* and let A and B be any two points on *l* (Incidence Axiom 2). (See Figure 3.23.)

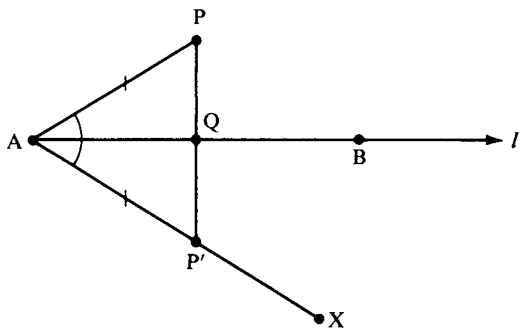


FIGURE 3.23

(2) On the opposite side of l from P there exists a ray $\overrightarrow{AX}$ such that $\sphericalangle XAB \cong \sphericalangle PAB$ (Congruence Axiom 4).
(3) There is a point P′ on $\overrightarrow{AX}$ such that $AP' \cong AP$ (Congruence Axiom 1).
(4) PP′ intersects l in a point Q (definition of opposite sides of l).
(5) If Q = A, then $\overleftrightarrow{PP'} \perp l$ (definition of $\perp$).
(6) If Q ≠ A, then $\triangle PAQ \cong \triangle P'AQ$ (SAS).
(7) Hence, $\sphericalangle PQA \cong \sphericalangle P'QA$ (corresponding angles), so $\overleftrightarrow{PP'} \perp l$ (definition of $\perp$).
(8) Assume now that P lies on l. Since there are points not lying on l (Proposition 2.3), we can drop a perpendicular from one of them to l (steps 5 and 7), thereby obtaining a right angle.
(9) We can lay off an angle congruent to this right angle with vertex at P and one side on l (Congruence Axiom 4); the other side of this angle is part of a line through P perpendicular to l (Proposition 3.15(b)). ■

It is natural to ask whether the perpendicular to l through P constructed in Proposition 3.16 is unique. If P lies on l, Proposition 3.23 (later in this chapter) and the uniqueness part of Congruence Axiom 4 guarantee that the perpendicular is unique. If P does not lie on l, we will not be able to prove uniqueness for the perpendicular until the next chapter.

Note on Elliptic Geometry. Informally, elliptic geometry may be thought of as the geometry on a Euclidean sphere with antipodal points identified (the model of incidence geometry first described

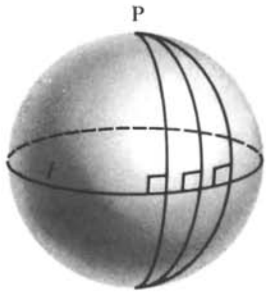


FIGURE 3.24

in Exercise 9(e), Chapter 2). Its "lines" are the great circles on the sphere. Given such a "line" l, there is a point P called the "pole" of l such that every line through P is perpendicular to l! To visualize this, think of l as the equator on a sphere and P as the north pole; every great circle through the north pole is perpendicular to the equator (Figure 3.24).

PROPOSITION 3.17 (ASA Criterion for Congruence). Given $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $AC \cong DF$. Then $\triangle ABC \cong \triangle DEF$.

PROPOSITION 3.18 (Converse of Proposition 3.10). If in $\triangle ABC$ we have $\sphericalangle B \cong \sphericalangle C$, then $AB \cong AC$ and $\triangle ABC$ is isosceles.

PROPOSITION 3.19 (Angle Addition). Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$, and $\sphericalangle GBA \cong \sphericalangle HED$. Then $\sphericalangle ABC \cong \sphericalangle DEF$. (See Figure 3.25.)

Proof:

(1) By the crossbar theorem,[3] we may assume G is chosen so that $A * G * C$.

(2) By Congruence Axiom 1, we assume D, F, and H chosen so that $AB \cong ED$, $GB \cong EH$, and $CB \cong EF$.

(3) Then $\triangle ABG \cong \triangle DEH$ and $\triangle GBC \cong \triangle HEF$ (SAS).

[3] This renaming technique will be used frequently. G is just a label for any point $\neq$ B on the ray which intersects AC, so we may as well choose G to be the point of intersection rather than clutter the argument with a new label.

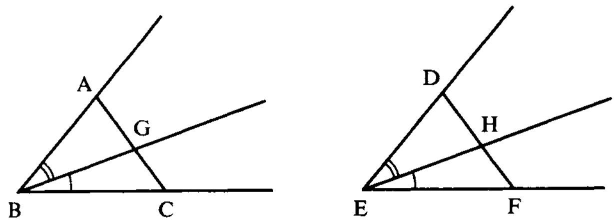


FIGURE 3.25

(4) $\sphericalangle DHE \cong \sphericalangle AGB$, $\sphericalangle FHE \cong \sphericalangle CGB$ (step 3), and $\sphericalangle AGB$ is supplementary to $\sphericalangle CGB$ (step 1).
(5) D, H, F are collinear and $\sphericalangle DHE$ is supplementary to $\sphericalangle FHE$ (step 4, Proposition 3.14, and Congruence Axiom 4).
(6) $D * H * F$ (Proposition 3.7, using the hypothesis on $\overrightarrow{EH}$).
(7) $AC \cong DF$ (steps 3 and 6, Congruence Axiom 3).
(8) $\sphericalangle BAC \cong \sphericalangle EDF$ (steps 3 and 6).
(9) $\triangle ABC \cong \triangle DEF$ (SAS; steps 2, 7, and 8).
(10) $\sphericalangle ABC \cong \sphericalangle DEF$ (corresponding angles). ■

PROPOSITION 3.20 (Angle Subtraction). Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$, and $\sphericalangle ABC \cong \sphericalangle DEF$. Then $\sphericalangle GBA \cong \sphericalangle HED$.

DEFINITION. $\sphericalangle ABC < \sphericalangle DEF$ means there is a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\sphericalangle ABC \cong \sphericalangle GEF$ (see Figure 3.26).

PROPOSITION 3.21 (Ordering of Angles). (a) Exactly one of the following conditions holds (*trichotomy*): $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle Q < \sphericalangle P$. (b) If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$. (c) If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P > \sphericalangle R$. (d) If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.

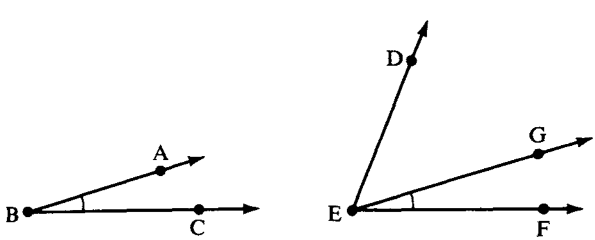


FIGURE 3.26

PROPOSITION 3.22 (SSS Criterion for Congruence). Given ΔABC and ΔDEF. If $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\Delta ABC \cong \Delta DEF$.

The AAS criterion for congruence will be given in the next chapter because its proof is more difficult. The next proposition was assumed as an axiom by Euclid, but can be proved from Hilbert's axioms.

PROPOSITION 3.23 (Euclid's Fourth Postulate). All right angles are congruent to each other. (See Figure 3.27.)

Proof:

(1) Given $\sphericalangle BAD \cong \sphericalangle CAD$ and $\sphericalangle FEH \cong \sphericalangle GEH$ (two pairs of right angles, by definition). Assume the contrary, that $\sphericalangle BAD$ is not congruent to $\sphericalangle FEH$ (RAA hypothesis).
(2) Then one of these angles is smaller than the other, e.g., $\sphericalangle FEH < \sphericalangle BAD$ (Proposition 3.21 (a)), so that by definition there is a ray $\overrightarrow{AJ}$ between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ such that $\sphericalangle BAJ \cong \sphericalangle FEH$.
(3) $\sphericalangle CAJ \cong \sphericalangle GEH$ (Proposition 3.14).
(4) $\sphericalangle CAJ \cong \sphericalangle FEH$ (steps 1 and 3, Congruence Axiom 5).
(5) There is a ray $\overrightarrow{AK}$ between $\overrightarrow{AD}$ and $\overrightarrow{AC}$ such that $\sphericalangle BAJ \cong \sphericalangle CAK$ (step 1 and Proposition 3.21 (b)).
(6) $\sphericalangle BAJ \cong \sphericalangle CAJ$ (steps 2 and 4, and Congruence Axiom 5).
(7) $\sphericalangle CAJ \cong \sphericalangle CAK$ (steps 5 and 6, and Congruence Axiom 5).
(8) Thus, we have $\sphericalangle CAD$ greater than $\sphericalangle CAK$ (by definition) and less than its congruent angle $\sphericalangle CAJ$ (step 7 and Proposition 3.8 (c)), which contradicts Proposition 3.21.
(9) $\sphericalangle BAD \cong \sphericalangle FEH$ (RAA conclusion). ■

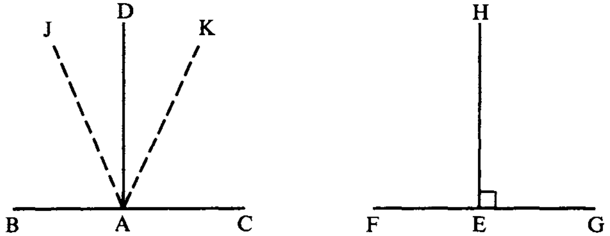


FIGURE 3.27

AXIOMS OF CONTINUITY

The axioms of continuity are needed to fill a number of gaps in Euclid's *Elements*. Consider the argument Euclid gives to justify his very first proposition.

EUCLID'S PROPOSITION 1. Given any segment, there is an equilateral triangle having the given segment as one of its sides.

Euclid's Proof:

(1) Let AB be the given segment. With center A and radius AB, let the circle BCD be described (Postulate III). (See Figure 3.28.)

(2) Again with center B and radius BA, let the circle ACE be described (Postulate III).

(3) From a point C in which the circles cut one another, draw the segments CA and CB (Postulate I).

(4) Since A is the center of the circle CDB, AC is congruent to AB (definition of circle).

(5) Again, since B is the center of circle CAE, BC is congruent to BA (definition of circle).

(6) Since CA and CB are each congruent to AB (steps 4 and 5), they are congruent to each other (first common notion).

(7) Hence, $\triangle ABC$ is an equilateral triangle (by definition) having AB as one of its sides. ■

Since very step has apparently been justified, you may not see the gap in the proof. It occurs in the first three steps, especially in the third step, which explicitly states that C is a point in which the circles cut

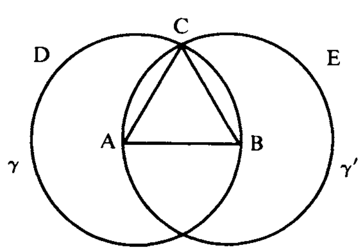


FIGURE 3.28

each other. (The second step states this implicitly by using the same letter "C" to denote part of the circle, as in the first step.) The point is: How do we know that such a point C exists?

If you believe it is obvious from the diagram that such a point C exists you are right—but you are not allowed to use the diagram to justify this! We aren't saying that the circles constructed do not cut each other; we're saying only that another axiom is needed to *prove* that they do.

The gap can be filled by assuming the following *circular continuity principle:*

CIRCULAR CONTINUITY PRINCIPLE. If a circle γ has one point inside and one point outside another circle γ', then the two circles intersect in two points.

Here a point P is defined as *inside* a circle with center O and radius OR if OP < OR (*outside* if OP > OR). In Figure 3.28, point B is inside circle γ', and the point B′ (not shown) such that A is the midpoint of BB′ is outside γ'. This principle is also needed to prove Euclid's 22nd proposition, the converse to the triangle inequality (see Major Exercise 4, Chapter 4). Another gap occurs in Euclid's method of dropping a perpendicular to a line (his 12th proposition, our Proposition 3.16). His construction tacitly assumes that if a line passes through a point inside a circle, then the line intersects the circle in two points—an assumption you can justify using the circular continuity principle (Major Exercise 1, Chapter 4; but our justification uses Proposition 3.16, so Euclid's argument must be discarded to avoid circular reasoning). Here is another useful consequence (see Major Exercise 2, Chapter 4).

ELEMENTARY CONTINUITY PRINCIPLE. If one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.

Can you see why these are "continuity principles"? For example, in Figure 3.29, if you were drawing the segment with a pencil moving continuously from A to B, it would have to cross the circle (if it didn't, there would be "a hole" in the segment or the circle).

The next statement is not about continuity but rather about measurement. Archimedes was astute enough to recognize that a new

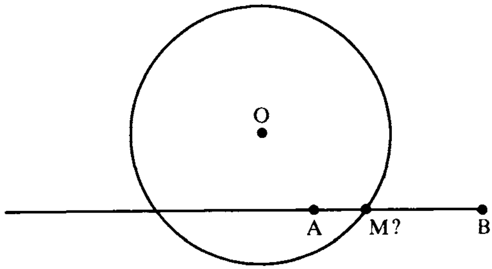


FIGURE 3.29

axiom was needed. It is listed here because we will show that it is a consequence of Dedekind's continuity axiom, given later in this section. It is needed so that we can assign a positive real number as the *length* $\overline{AB}$ of an arbitrary segment AB, as will be explained in Chapter 4.

ARCHIMEDES' AXIOM. If CD is any segment, A any point, and r any ray with vertex A, then for every point B $\neq$ A on r there is a number n such that when CD is laid off n times on r starting at A, a point E is reached such that $n \cdot CD \cong AE$ and either B = E or B is between A and E.

Here we use Congruence Axiom 1 to begin laying off CD on r starting at A, obtaining a unique point A_1 on r such that $AA_1 \cong CD$, and we define $1 \cdot CD$ to be AA_1. Let r_1 be the ray emanating from A_1 that is contained in r. By the same method, we obtain a unique point A_2 on r_1 such that $A_1A_2 \cong CD$, and we define $2 \cdot CD$ to be AA_2. Iterating this process, you can define, by induction on n, the segment $n \cdot CD$ to be AA_n.

For example, if AB were π units long and CD of one unit length, you would have to lay off CD at least four times to get to a point E beyond the point B (see Figure 3.30).

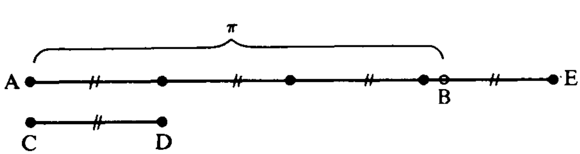


FIGURE 3.30

The intuitive content of Archimedes' axiom is that if you arbitrarily choose one segment CD as a unit of length, then every other segment has finite length with respect to this unit (in the notation of the axiom the length of AB with respect to CD as unit is at most n units). Another way to look at it is to choose AB as unit of length. The axiom says that no other segment can be infinitesimally small with respect to this unit (the length of CD with respect to AB as unit is at least $1/n$ units).

The next statement is a consequence of Archimedes' axiom and the previous axioms (as you will show in Exercise 6, Chapter 5), but if one wants to do geometry with segments of infinitesimal length allowed, this statement can replace Archimedes' axiom (see my note "Aristotle's Axiom in the Foundations of Hyperbolic Geometry," *Journal of Geometry*, vol. 33, 1988). Besides, Archimedes' axiom is not a purely geometric axiom, since it asserts the existence of a *number*.

ARISTOTLE'S AXIOM. Given any side of an acute angle and any segment AB, there exists a point Y on the given side of the angle such that if X is the foot of the perpendicular from Y to the other side of the angle, XY > AB.

Informally, if we start with any point Y on the given side, then as Y "recedes endlessly" from the vertex V of the angle, perpendicular segment XY "increases indefinitely" (because it is eventually bigger than any previously given segment AB). This principle will be valuable in Chapter 5 when we examine Proclus' attempt to prove Euclid's parallel postulate (see Figure 5.2). The idea of the proof from Archimedes' axiom is that if the starting XY is not already greater than the given segment AB, one simply lays off enough copies of VY on ray $\overrightarrow{VY}$ until point Y′ is reached such that the perpendicular segment dropped from Y′ is greater than AB (see Exercise 6, Chapter 5).

IMPORTANT COROLLARY. Let $\overrightarrow{AB}$ be any ray, P any point not collinear with A and B, and $\sphericalangle$XVY any acute angle. Then there exists a point R on ray $\overrightarrow{AB}$ such that $\sphericalangle PRA < \sphericalangle XVY$.

Informally, if we start with any point R on $\overrightarrow{AB}$, then as R "recedes endlessly" from the vertex A of the ray, $\sphericalangle$PRA decreases to zero (because it is eventually smaller than any previously given angle $\sphericalangle$XVY). This result will be used in Chapter 6. Its proof uses Theorem 4.2 of Chapter 4 (the exterior angle theorem) and so it should be given

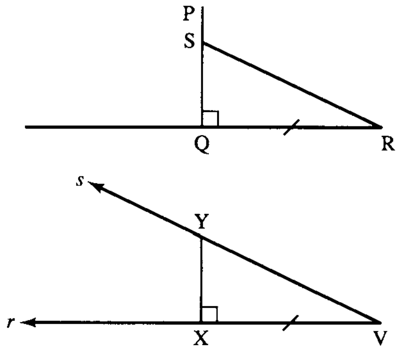


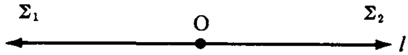
FIGURE 3.31

after that theorem is proved, but we sketch the proof now for convenience of reference. You may skip it now and return when needed.

Proof:
Let Q be the foot of the perpendicular from P to $\overleftrightarrow{AB}$. Since point B is just a label, we choose it so that Q $\neq$ B and Q lies on ray $\overrightarrow{BA}$. X and Y are arbitrary points on the rays *r* and *s* that are the sides of $\sphericalangle$XVY. Let X′ be the foot of the perpendicular from Y to the line containing *r*. By the hypothesis that the angle is acute and the exterior angle theorem, we can show (by an RAA argument) that X′ actually lies on *r*, and so we can choose X to be X′. Aristotle's axiom guarantees that Y can be chosen such that XY > PQ. By Congruence Axiom 1, there is one point R on $\overrightarrow{QB}$ such that QR $\cong$ XV. We claim that $\sphericalangle$PRQ < $\sphericalangle$XVY. Assume the contrary. By trichotomy, there is a ray $\overrightarrow{RS}$ such that $\sphericalangle$QRS $\cong$ $\sphericalangle$XVY and $\overrightarrow{RS}$ either equals $\overrightarrow{RP}$ or is between $\overrightarrow{RP}$ and $\overrightarrow{RQ}$. By the crossbar theorem, point S (which thus far is also merely a label) can be chosen to lie on segment PQ; then SQ is not greater than PQ. By the ASA congruence criterion, SQ $\cong$ XY. Hence XY is not greater than PQ, contradicting our choice of Y. Thus $\sphericalangle$PRQ < $\sphericalangle$XVY, as claimed. If R lies on ray $\overrightarrow{AB}$, then $\sphericalangle$PRQ = $\sphericalangle$PRA and we are done. If not, R and Q lie on the opposite ray. By the exterior angle theorem, if R′ is any point such that Q * R * R′, then $\sphericalangle$PR′Q < $\sphericalangle$PRQ < XVY. We get $\sphericalangle$PBA = $\sphericalangle$PBQ < $\sphericalangle$XVY by taking R′ = B. ■

All four principles thus far stated are in the spirit of ancient Greek geometry. They are all consequences of the next axiom, which is utterly modern.

FIGURE 3.32



DEDEKIND'S AXIOM.[4] Suppose that the set $\{l\}$ of all points on a line l is the disjoint union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of either subset is between two points of the other. Then there exists a unique point O on l such that one of the subsets is equal to a ray of l with vertex O and the other subset is equal to the complement.

Dedekind's axiom is a sort of converse to the line separation property stated in Proposition 3.4. That property says that any point O on l separates all the other points on l into those to the left of O and those to the right (see Figure 3.32; more precisely, $\{l\}$ is the union of the two rays of l emanating from O). Dedekind's axiom says that, conversely, any separation of points on l into left and right is produced by a unique point O. A pair of subsets Σ_1 and Σ_2 with the properties in Dedekind's axiom is called a *Dedekind cut* of the line.

Loosely speaking, the purpose of Dedekind's axiom is to ensure that a line l has no "holes" in it, in the sense that for any point O on l and any positive real number x there exist unique points P_{-x} and P_x on l such that $P_{-x} * O * P_x$ and segments $P_{-x}O$ and OP_x both have length x (with respect to some unit segment of measurement); see Figure 3.33.

Without Dedekind's axiom there would be no guarantee, for example, of the existence of a segment of length π. With it, we can introduce a rectangular coordinate system into the plane and do geometry analytically, as Descartes and Fermat discovered in the seventeenth century. This coordinate system enables us to prove that our axioms for Euclidean geometry are *categorical* in the sense that the system has a unique model (up to isomorphism — see the section Isomorphism of Models in Chapter 2), namely, the usual Cartesian coordinate plane of all ordered pairs of real numbers.

If we omitted Dedekind's axiom, then another model would be the so-called *surd plane,* a plane that is used to prove the impossibility of

[4] This axiom was proposed by J. W. R. Dedekind in 1871; an analogue of it is used in analysis texts to express the completeness of the real number system. It implies that every Cauchy sequence converges, that continuous functions satisfy the intermediate value theorem, that the definite integral of a continuous function exists, and other important conclusions. Dedekind actually defined a "real number" as a Dedekind cut on the set of rational numbers, an idea Eudoxus had 2000 years earlier (see Moise, 1990, Chapter 20).

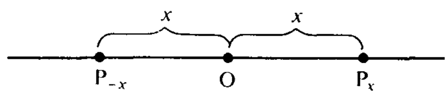


FIGURE 3.33

trisecting every angle with a straightedge and compass (see Moise, 1990, p. 282 ff.). The categorical natural of all the axioms is proved in Borsuk and Szmielew (1960, p. 276 ff.).

Warning. If you have never seen Dedekind's axiom before, arguments using it may be difficult to follow. Don't be discouraged. With the exception of Theorem 6.6 in hyperbolic geometry, it is not needed for studying the main theme of this book. I advise the beginning student to skip to the next section, Axiom of Parallelism.

Although Dedekind's axiom implies the other four principles and is the only continuity axiom we need assume, we still refer to the others as "axioms." Let us sketch a proof that *Archimedes' axiom is a consequence of Dedekind's* (and the axioms preceding this section).

Proof:
Given a segment CD and a point A on line l, with a ray r of l emanating from A. In the terminology of Archimedes' axiom, let Σ_1 consist of A and all points B on r reached by laying off copies of segment CD on r starting from A. Let Σ_2 be the complement of Σ_1 in r. We wish to prove that Σ_2 is empty, so assume the contrary.

In that case, let us show that we have defined a Dedekind cut of r (see Exercise 7(a)). Start with two points P, Q in Σ_2 and say $A * P * Q$. We must show that $PQ \subset \Sigma_2$. Let B be between P and Q. Suppose B could be reached, so that n and E are as in the statement of Archimedes' axiom; then, by Proposition 3.3, P is reached by the same n and E, contradicting $P \in \Sigma_2$. Thus $PQ \subset \Sigma_2$. Similarly, you can show that when P and Q are two points in Σ_1, $PQ \subset \Sigma_1$ (Exercise 7(b)). So we have a Dedekind cut. Let O be the point of r furnished by Dedekind's axiom.

Case 1. $O \in \Sigma_1$. Then for some number n, O can be reached by laying off n copies of segment CD on r starting from A. By laying off one more copy of CD, we can reach a point in Σ_2, but by definition of Σ_2, that is impossible.

Case 2. $O \in \Sigma_2$. Lay off a copy of CD on the ray opposite to Σ_2 starting at O, obtaining a point P; P lies on r (Exercise 7(b)), so $P \in \Sigma_1$. Then for some number n, P can be reached by laying off n copies of segment CD on r starting from A. By laying off one more copy of CD, we can reach O. That contradicts $O \in \Sigma_2$.

So in either case, we obtain a contradiction, and we can reject the RAA hypothesis that Σ_2 is nonempty. ■

To further get an idea of how Dedekind's axiom gives us continuity results, we sketch a proof now of the elementary continuity principle from Dedekind's axiom (logically, this proof should be given later, because it uses results from Chapter 4). Refer to Figure 3.29, p. 95.

Proof:
By the definitions of "inside" and "outside" of a circle γ with center O and radius OR, we have $OA < OR < OB$. Let Σ_2 be the set of all points P on the ray $\overrightarrow{AB}$ that either lie on γ or are outside γ, and let Σ_1 be its complement in $\overrightarrow{AB}$. By trichotomy (Proposition 3.13(a)), Σ_1 consists of all points of the segment AB that lie inside γ. Applying Exercise 27 of Chapter 4, you can convince yourself that (Σ_1, Σ_2) is a Dedekind cut. Let M be the point on $\overrightarrow{AB}$ furnished by Dedekind's axiom. Assume M does not lie on γ (RAA hypothesis).

Case 1. $OM < OR$. Then $M \in \Sigma_1$. Let m and r be the lengths (defined in Chapter 4) of OM and OR, respectively. Since Σ_2 with M is a ray, there is a point $N \in \Sigma_2$ such that the length of MN is $\frac{1}{2}(r-m)$ (e.g., by laying off a segment whose length is $\frac{1}{2}(r-m)$, using Theorem 4.3(11)). But by the *triangle inequality* (Corollary 2 to Theorem 4.3), the length of ON is less than $m+\frac{1}{2}(r-m) < m+(r-m) = r$, which contradicts $N \in \Sigma_2$.

Case 2. OM > OR. The same argument applies, interchanging the roles of Σ_2 and Σ_1.

So in either case, we obtain a contradiction, and M must lie on γ. ■

You will find a lovely proof of the circular continuity principle from Dedekind's axiom on pp. 238–240 of Heath's translation and commentary on Euclid's *Elements* (1956). It assumes that Dedekind's axiom holds for semicircles, which you can easily prove in Major Exercise 4, and also uses the triangle inequality and the fact that the hypotenuse is greater than the leg (proved in Chapter 4).

Euclid's tacit use of continuity principles can often be avoided. We did not use them in our proof of the existence of perpendiculars (Proposition 3.16). We did use the circular continuity principle to prove the existence of equilateral triangles on a given base, and Euclid used that to prove the existence of midpoints, as in your straightedge-and-compass solution to Major Exercise 1 (a) of Chapter 1. But there is an ingenious way to prove the existence of midpoints using only the very mild continuity given by Pasch's theorem (see Exercise 12, Chapter 4).

Figure 3.34 shows the implications discussed (assuming all the incidence, betweenness, and congruence axioms—especially SAS).

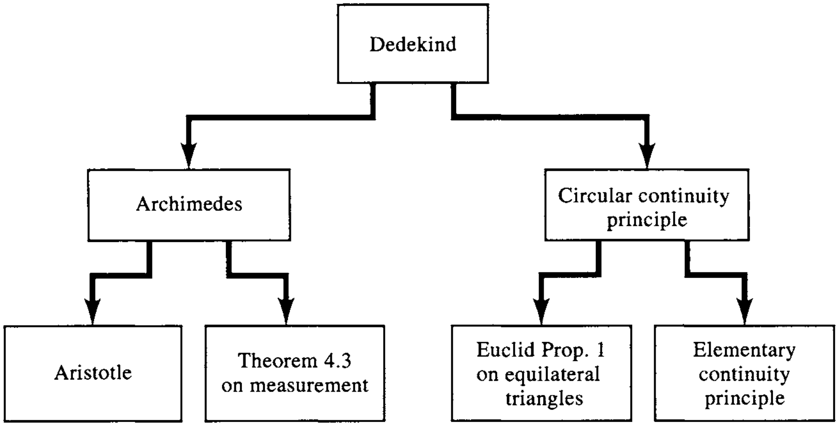


FIGURE 3.34

AXIOM OF PARALLELISM

If we were to stop with the axioms we now have, we could do quite a bit of geometry, but we still couldn't do all of Euclidean geometry. We would be able to do what J. Bolyai called "absolute geometry." This name is misleading because it does not include elliptic geometry and other geometries (see Appendix B). Preferable is the name suggested by W. Prenowitz and M. Jordan (1965), *neutral geometry,* so called because in doing this geometry we remain neutral about the one axiom from Hilbert's list left to be considered — historically the most controversial axiom of all.

HILBERT'S AXIOM OF PARALLELISM. For every line *l* and every point P not lying on *l* there is at most one line *m* through P such that *m* is parallel to *l* (Figure 3.35).

Note that this axiom is weaker than the Euclidean parallel postulate introduced in Chapter 1. This axiom asserts only that *at most* one line through P is parallel to *l*, whereas the Euclidean parallel postulate asserts in addition that *at least* one line through P is parallel to *l*. The reason "at least" is omitted from Hilbert's axiom is that it can be proved from the other axioms (see Corollary 2 to Theorem 4.1 in Chapter 4); it is therefore unnecessary to assume this as part of an axiom. This observation is important because it implies that the elliptic parallel property (no parallel lines exist) is inconsistent with the axioms of neutral geometry. Thus, a different set of axioms is needed for the foundation of elliptic geometry (see Appendix A).

The axiom of parallelism completes our list of 16 axioms for Euclidean geometry. A *Euclidean plane* is a model of these axioms. In referring to these axioms we will use the following shorthand: the incidence axioms will be denoted by I-1, I-2, and I-3; the betweenness axioms by B-1, B-1, B-3, and B-4; the congruence axioms by C-1, C-2, C-3, C-4, C-5, and C-6 (or SAS). The continuity axioms and the parallelism axiom will be referred to by name.

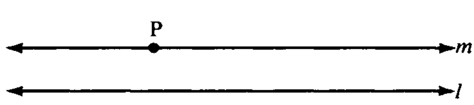


FIGURE 3.35

REVIEW EXERCISE

Which of the following statements are correct?

(1) Hilbert's axiom of parallelism is the same as the Euclidean parallel postulate given in Chapter 1.
(2) A * B * C is logically equivalent to C * B * A.
(3) In Axiom B-2 it is unnecessary to assume the existence of a point E such that B * D * E because this can be proved from the rest of the axiom and Axiom B-1, by interchanging the roles of B and D and taking E to be A.
(4) If A, B, and C are distinct collinear points, it is possible that *both* A * B * C *and* A * C * B.
(5) The "line separation property" asserts that a line has two sides.
(6) If points A and B are on opposite sides of a line l, then a point C not on l must be either on the same side of l as A or on the same side of l as B.
(7) If line m is parallel to line l, then all the points on m lie on the same side of l.
(8) If we were to take Pasch's theorem as an axiom instead of the separation axiom B-4, then B-4 could be proved as a theorem.
(9) The notion of "congruence" for two triangles is not defined in this chapter.
(10) It is an immediate consequence of Axiom C-2 that if AB ≅ CD, then CD ≅ AB.
(11) One of the congruence axioms asserts that if congruent segments are "subtracted" from congruent segments, the differences are congruent.
(12) In the statement of Axiom C-4 the variables A, B, C, A′, and B′ are quantified universally, and the variable C′ is quantified existentially.
(13) One of the congruence axioms is the side-side-side (SSS) criterion for congruence of triangles.
(14) Euclid attempted unsuccessfully to prove the side-angle-side criterion (SAS) for congruence by a method called "superposition."
(15) We can use Pappus' method to prove the converse of the theorem on base angles of an isosceles triangle if we first prove the angle-side-angle (ASA) criterion for congruence.
(16) Archimedes' axiom is independent of the other 15 axioms for Euclidean geometry given in this book.
(17) AB < CD means that there is a point E between C and D such that AB ≅ CE.
(18) Neutral geometry used to be called *absolute geometry;* it is the geometry you have when the axiom of parallelism is excluded from the system of axioms given here.

EXERCISES ON BETWEENNESS

1. Given A * B * C and A * C * D.
   (a) Prove that A, B, C, and D are four distinct points (the proof requires an axiom).
   (b) Prove that A, B, C, and D are collinear.
   (c) Prove the corollary to Axiom B-4.
2. (a) Finish the proof of Proposition 3.1 by showing that $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.
   (b) Finish the proof of Proposition 3.3 by showing that A * B * D.
   (c) Prove the converse of Proposition 3.3 by applying Axiom B-1.
   (d) Prove the corollary to Proposition 3.3.
3. Given A * B * C.
   (a) Use Proposition 3.3 to prove that AB ⊂ AC. Interchanging A and C, deduce CB ⊂ CA; which axiom justifies this interchange?
   (b) Use Axiom B-4 to prove that AC ⊂ AB ∪ BC. (Hint: If P is a fourth point on AC, use another line through P to show P ∈ AB or P ∈ BC.)
   (c) Finish the proof of Proposition 3.5. (Hint: If P ≠ B and P ∈ AB ∩ BC, use another line through P to get a contradiction.)
4. Given A * B * C.
   (a) If P is a fourth point collinear with A, B, and C, use Proposition 3.3 and an axiom to prove that ~A * B * P ⇒ ~A * C * P.
   (b) Deduce that $\overrightarrow{BA} \subset \overrightarrow{CA}$ and, symmetrically, $\overrightarrow{BC} \subset \overrightarrow{AC}$.
   (c) Use this result, Proposition 3.1 (a), Proposition 3.3, and Proposition 3.5 to prove that B is the only point that $\overrightarrow{BA}$ and $\overrightarrow{BC}$ have in common.
5. Given A * B * C. Prove that $\overrightarrow{AB} = \overrightarrow{AC}$, completing the proof of Proposition 3.6. Deduce that every ray has a *unique* opposite ray.
6. In Axiom B-2 we were given distinct points B and D and we asserted the existence of points A, C, and E such that A * B * D, B * C * D, and B * D * E. We can now show that it was not necessary to assume the existence of a point C between B and D because we can prove from our other axioms (including the rest of Axiom B-2) and from Pasch's theorem (which was proved without using Axiom B-2) that C exists.[5] Your job is to justify each step in the proof (some of the steps require a separate RAA argument).

[5] Regarding superfluous hypotheses, there is a story that Napoleon, after examining a copy of Laplace's *Celestial Mechanics*, asked Laplace why there was no mention of God in the work. The author replied, "I have no need of this hypothesis."

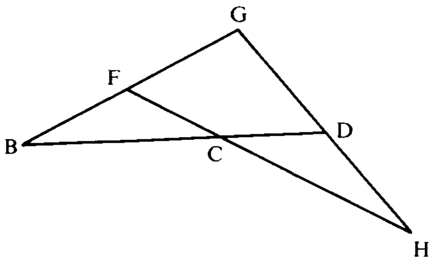


FIGURE 3.36

Proof (see Figure 3.36):

(1) There exists a line $\overrightarrow{BD}$ through B and D.
(2) There exists a point F not lying on $\overrightarrow{BD}$.
(3) There exists a line $\overrightarrow{BF}$ through B and F.
(4) There exists a point G such that B $*$ F $*$ G.
(5) Points B, F, and G are collinear.
(6) G and D are distinct points and D, B, and G are not collinear.
(7) There exists a point H such that G $*$ D $*$ H.
(8) There exists a line $\overrightarrow{GH}$.
(9) H and F are distinct points.
(10) There exists a line $\overleftrightarrow{FH}$.
(11) D does not lie on $\overleftrightarrow{FH}$.
(12) B does not lie on $\overleftrightarrow{FH}$.
(13) G does not lie on $\overleftrightarrow{FH}$.
(14) Points D, B, and G determine $\triangle$DBG and $\overleftrightarrow{FH}$ intersects side BG in a point between B and G.
(15) H is the only point lying on both $\overleftrightarrow{FH}$ and $\overleftrightarrow{GH}$.
(16) No point between G and D lies on $\overleftrightarrow{FH}$.
(17) Hence, $\overleftrightarrow{FH}$ intersects side BD in a point C between D and B.
(18) Thus, there exists a point C between D and B. ■

7. (a) Define a Dedekind cut on a ray *r* the same way a Dedekind cut is defined for a line. Prove that the conclusion of Dedekind's axiom also holds for *r*. (Hint: One of the subsets, say, Σ_1, contains the vertex A of *r*; enlarge this set so as to include the ray opposite to *r* and show that a Dedekind cut of the line *l* containing *r* is obtained.) Similarly, state and prove a version of Dedekind's axiom for a cut on a segment.
   (b) Supply the indicated arguments left out of the proof of Archimedes' axiom from Dedekind's axiom.
8. From the three-point model (Example 1 in Chapter 2) we saw that if we used only the axioms of incidence we could not prove that a line has

more than two points lying on it. Using the betweenness axioms as well, prove that every line has at least five points lying on it. Give an informal argument to show that every segment (a fortiori, every line) has an infinite number of points lying on it (a formal proof requires the technique of mathematical induction).

9. Given a line l, a point A on l, and a point B not on l. Then every point of the ray $\overrightarrow{AB}$ (except A) is on the same side of l as B. (Hint: Use an RAA argument).
10. Prove Proposition 3.7.
11. Prove Proposition 3.8. (Hint: For Proposition 3.8(c) prove in two steps that E and B lie on the same side of $\overleftrightarrow{AD}$, first showing that EB does not meet $\overrightarrow{AD}$, then showing that EB does not meet the opposite ray $\overrightarrow{AF}$. Use Exercise 9.)
12. Prove the crossbar theorem. (Hint: Assume the contrary, and show that B and C lie on the same side of $\overleftrightarrow{AD}$. Use Proposition 3.8(c) to derive a contradiction.)
13. Prove Proposition 3.9. (Hint: For Proposition 3.9(a) use Pasch's theorem and Proposition 3.7; see Figure 3.37. For Proposition 3.9(b) let the ray emanate from point D in the interior of $\triangle$ABC. Use the crossbar theorem and Proposition 3.7 to show that $\overrightarrow{AD}$ meets BC in a point E such that A * D * E. Apply Pasch's theorem to $\triangle$ABE and $\triangle$AEC; see Figure 3.38.)
14. Prove that a line cannot be contained in the interior of a triangle.
15. If a, b, and c are rays, let us say that they are *coterminal* if they emanate from the same point, and let us use the notation $a * b * c$ to mean that b is between a and c (as defined on p. 82). The analogue of Axiom B-1 states that if $a * b * c$, then a, b, c are distinct and coterminal and $c * b * a$; this analogue is obviously correct. State the analogues of Axioms B-2 and B-3 and Proposition 3.3 and tell which parts of these analogues are correct. (Beware of opposite rays!)
16. Find an interpretation in which the incidence axioms and the first two betweenness axioms hold but Axiom B-3 fails in the following way: there

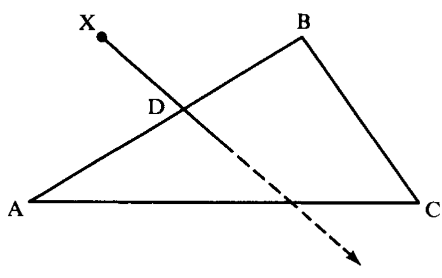


FIGURE 3.37

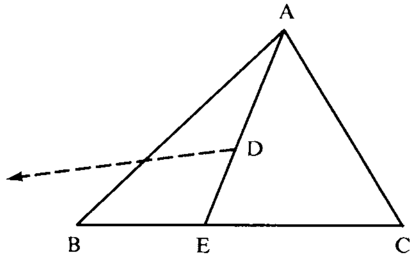


FIGURE 3.38

exist three collinear points, no one of which is between the other two. (Hint: In the usual Euclidean model, introduce a new betweenness relation A * B * C to mean that B is the midpoint of AC.)

17. Find an interpretation in which the incidence axioms and the first three betweenness axioms hold but the line separation property (Proposition 3.4) fails. (Hint: In the usual Euclidean model, pick a point P that is between A and B in the usual Euclidean sense and specify that A will now be considered to be between P and B. Leave all other betweenness relations among points alone. Show that P lies neither on ray $\overrightarrow{AB}$ nor on its opposite ray $\overrightarrow{AC}$.)
18. A rational number of the form $a/2^n$ (with a, n integers) is called *dyadic.* In the interpretations of Project 2 for this chapter, restrict to those points which have dyadic coordinates and to those lines which pass through several dyadic points. The incidence axioms, the first three betweenness axioms, and the line separation property all hold in this dyadic rational plane; show that Pasch's theorem fails. (Hint: The lines $3x + y = 1$ and $y = 0$ do not meet in this plane.)
19. A set of points S is called *convex* if whenever two points A and B are in S, the entire segment AB is contained in S. Prove that a half-plane, the interior of an angle, and the interior of a triangle are all convex sets, whereas the exterior of a triangle is not convex. Is a triangle a convex set?

EXERCISES ON CONGRUENCE

20. Justify each step in the following proof of Proposition 3.11:

Proof:

(1) Assume on the contrary that BC is not congruent to EF.
(2) Then there is a point G on $\overrightarrow{EF}$ such that BC ≅ EG.

(3) $G \neq F$.
(4) Since $AB \cong DE$, adding gives $AC \cong DG$.
(5) However, $AC \cong DF$.
(6) Hence, $DF \cong DG$.
(7) Therefore, $F = G$.
(8) Our assumption has led to a contradiction; hence, $BC \cong EF$. ■

21. Prove Proposition 3.13(a). (Hint: In case AB and CD are not congruent, there is a unique point $F \neq D$ on $\overrightarrow{CD}$ such that $AB \cong CF$ (reason ?). In case $C * F * D$, show that $AB < CD$. In case $C * D * F$, use Proposition 3.12 and some axioms to show that $CD < AB$.)
22. Use Proposition 3.12 to prove Proposition 3.13(b) and (c).
23. Use the previous exercise and Proposition 3.3 to prove Proposition 3.13(d).
24. Justify each step in the following proof of Proposition 3.14 (see Figure 3.39).

Proof:
Given $\sphericalangle ABC \cong \sphericalangle DEF$. To prove $\sphericalangle CBG \cong \sphericalangle FEH$:
(1) The points A, C, and G being given arbitrarily on the sides of $\sphericalangle ABC$ and the supplement $\sphericalangle CBG$ of $\sphericalangle ABC$, we can choose the points D, F, and H on the sides of the other angle and its supplement so that $AB \cong DE$, $CB \cong FE$, and $BG \cong EH$.
(2) Then, $\triangle ABC \cong \triangle DEF$.
(3) Hence, $AC \cong DF$ and $\sphericalangle A \cong \sphericalangle D$.
(4) Also, $AG \cong DH$.
(5) Hence, $\triangle ACG \cong \triangle DFH$.
(6) Therefore, $CG \cong FH$ and $\sphericalangle G \cong \sphericalangle H$.
(7) Hence, $\triangle CBG \cong \triangle FEH$.
(8) It follows that $\sphericalangle CBG \cong \sphericalangle FEH$, as desired. ■

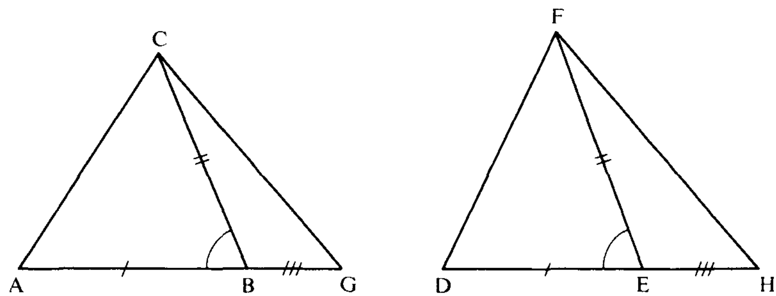


FIGURE 3.39

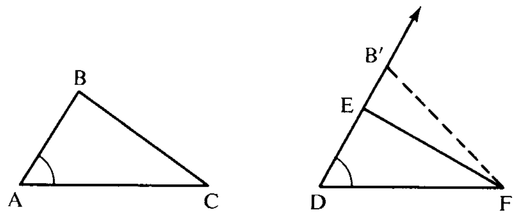


FIGURE 3.40

25. Deduce Proposition 3.15 from Proposition 3.14.
26. Justify each step in the following proof of Proposition 3.17 (see Figure 3.40):

    Proof:
    Given $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $AC \cong DF$. To prove $\triangle ABC \cong \triangle DEF$:
    (1) There is a unique point B' on ray $\overrightarrow{DE}$ such that $DB' \cong AB$.
    (2) $\triangle ABC \cong \triangle DB'F$.
    (3) Hence, $\sphericalangle DFB' \cong \sphericalangle C$.
    (4) This implies $\overrightarrow{FE} = \overrightarrow{FB'}$.
    (5) In that case, $B' = E$.
    (6) Hence, $\triangle ABC \cong \triangle DEF$. ■

27. Prove Proposition 3.18.
28. Prove that an equiangular triangle (all angles congruent to one another) is equilateral.
29. Prove Proposition 3.20. (Hint: Use Axiom C-4 and Proposition 3.19.)
30. Given $\sphericalangle ABC \cong \sphericalangle DEF$ and $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Prove that there is a unique ray $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\sphericalangle ABG \cong \sphericalangle DEH$. (Hint: Show that D and F can be chosen so that $AB \cong DE$ and $BC \cong EF$, and that G can be chosen so that $A * G * C$. Use Propositions 3.7 and 3.12 and SAS to get H; see Figure 3.25.)
31. Prove Proposition 3.21 (imitate Exercises 21–23).
32. Prove Proposition 3.22. (Hint: Use the corollary to SAS to reduce to the case where $A = D$, $C = F$, and the points B and E are on opposite sides of $\overleftrightarrow{AC}$. Then consider the three cases in Figure 3.41 separately.)
33. If $AB < CD$, prove that $2AB < 2CD$.
34. Let $\mathbb{Q}^2$ be the *rational plane* of all ordered pairs (x, y) of rational numbers with the usual interpretations of the undefined geometric terms used in analytic geometry. Show that Axiom C-1 and the elementary continuity principle fail in $\mathbb{Q}^2$. (Hint: The setgment from $(0, 0)$ to $(1, 1)$ cannot be laid off on the x axis from the origin.)

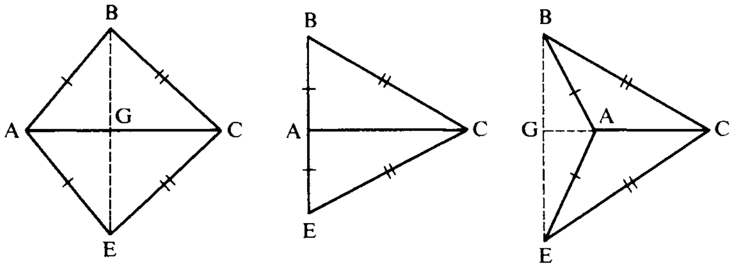


FIGURE 3.41

35. In the usual Euclidean plane we are all familiar with, there is a notion of length of a segment. Let us agree to measure all lengths in inches except for segments on one particular line called the x axis, where we will measure lengths in feet, and let us now interpret *congruence of segments* to mean that two segments have the same "length" in this perverse way of measuring. Incidence, betweenness, and congruence of angles will have their usual meaning. Show informally that the first five congruence axioms and angle addition (Proposition 3.19) still hold in this interpretation but that SAS fails (see Figure 3.42). Draw a picture of a "circle" with center on the x axis in this interpretation and use that picture to show that the circular continuity principle and the elementary continuity principle fail. Show that Dedekind's axiom still holds. Draw other pictures to show that SSS, ASA, and SAA all fail.
36. In Chapter 2 we displayed many models of the incidence axioms. As soon as we add the betweenness axioms, most of those interpretations are no longer models (for example, we lose all the finite models and the models in which "lines" are circles). Show, however, that the model in Exercise 9(d), which has the hyperbolic parallel property, is still a model under the natural interpretation of betweenness. It is called *the Klein model* and will be further studied in Chapter 7. Draw a picture to show that in this model, a point in the interior of an angle need not lie on a segment joining a point on one ray of the angle to a point on the other ray.

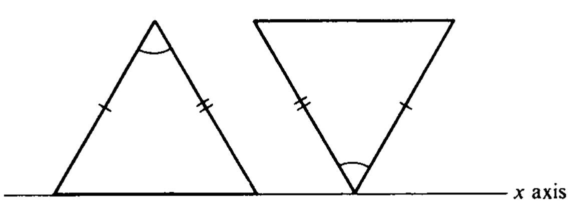


FIGURE 3.42

MAJOR EXERCISES

1. Let γ be a circle with center A and radius of length r. Let γ' be another circle with center A′ and radius of length r', and let d be the distance from A to A′ (see Figure 3.43). There is a hypothesis about the numbers r, r', and d that ensures that the circles γ and γ' intersect in two distinct points. Figure out what this hypothesis is. (Hint: It's statement that certain numbers obtained from r, r', and d are less than certain others.)·

   What hypothesis on r, r', and d ensures that γ and γ' intersect in only one point, i.e., that the circles are tangent to each other? (See Figure 3.44.)
2. Define the *reflection* in a line m to be the transformation R_m of the plane which leaves each point of m fixed and transforms a point A not on m as follows. Let M be the foot of the perpendicular from A to m. Then, by definition, $R_m(A)$ is the unique point A′ such that A′ * M * A and A′M ≅ MA. (See Figure 3.45.) This definition uses the result from Chapter 4 that the perpendicular from A to m is unique, so that the *foot* M is uniquely determined as the intersection with m. Prove that R_m is a *motion,* i.e., that AB ≅ A′B′ for any segment AB. Prove also that AB ≅ CD ⇒ A′B′ ≅ C′D′, and that ∢A ≅ ∢B ⇒ ∢A′ ≅ ∢B′. (Chapter 9 will be devoted to a thorough study of motions; the reflections generate the group of all such transformations.) (Hint: The proof breaks into the cases (i) A or B lies on m, (ii) A and B lie on opposite sides of m, and (iii) A and B

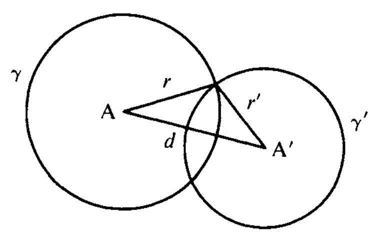


FIGURE 3.43

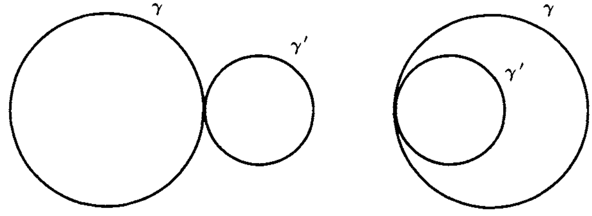


FIGURE 3.44

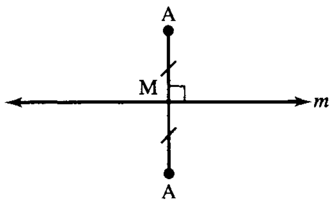


FIGURE 3.45

lie on the same side of *m*. In (ii), let M, N be the midpoints of AA′, BB′ and let C be the point at which AB meets *m*; prove that A′ * C * B′ by showing that ∢A′CM ≅ B′CN and apply Axiom C-3. In (iii), let C be the point at which AB′ meets *m*, and use B = (B′)′ and the first two cases to show that △ABC ≅ △A′B′C. Take care not to use results that are valid only in Euclidean geometry.)

Note. In elliptic geometry the perpendicular from A to *m* is unique except for one point P called the *pole* of *m* (see Figure 3.24, where *m* is the equator and P is the north pole); the definition of reflection is modified in elliptic geometry so that $R_m(P) = P$. Can you see that R_m is then the same as the 180° rotation about P? Recall that antipodal points are identified.

3. Consider the following statements on congruence:
   1. Given triangle △ABC and segment DE such that AB ≅ DE. Then on a given side of $\overleftrightarrow{DE}$ there is a unique point F such that AC ≅ DF and BC ≅ EF.
   2. Given triangles △ADC and △A′D′C′ and given A * B * C and A′ * B′ * C′. If AB ≅ A′B′, BC ≅ B′C′, AD ≅ A′D′, and BD ≅ B′D′, then CD ≅ C′D′ ("rigidity of a triangle with a tail"—see Figure 3.46).

   Prove these statements. Also, prove a statement 2a obtained from statement 2 by substituting CD ≅ C′D′ for BD ≅ B′D′ in the hypothesis and making BD ≅ B′D′ the conclusion.

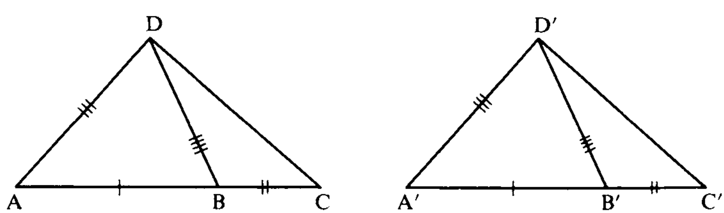


FIGURE 3.46

FIGURE 3.47

In Borsuk and Szmielew (1960), statements 1 and 2 are taken as axioms, in place of our Axioms C-4, C-5, and C-6. The advantage of this change is that these new congruence axioms refer only to congruence of segments. Congruence of angles, $\sphericalangle ABC \cong \sphericalangle A'B'C'$, can then be *defined* by specifying that A and C (respectively, A′ and C′) can be chosen on the sides of $\sphericalangle B$ (respectively, $\sphericalangle B'$) so that $AB \cong A'B'$, $BC \cong B'C'$, and $AC \cong A'C'$. With this definition, keeping the same incidence and betweenness axioms as before, show that C-4, C-5, and C-6 can be proved from C-1, C-2, C-3, and statements 1 and 2. (Hint: First prove statement 2a by an RAA argument. Then show that if $\sphericalangle ABC \cong \sphericalangle A'B'C'$, and that if we had chosen other points D, E, D′, and E′ on the sides of $\sphericalangle B$ and $\sphericalangle B'$ such that $DB \cong D'B'$ and $EB \cong E'B'$, then $DE \cong D'E'$. See Figure 3.47.)

4. Let AB be a diameter of circle γ with center O. The intersection σ of γ with one of the half-planes determined by $\overleftrightarrow{AB}$ is called an *open semicircle of* γ with endpoints A, B; adding the points A, B gives the *semicircle* $\bar{\sigma}$. Define a betweenness relation # on σ as follows: P # Q # R means that P, Q, and R are distinct points on σ and $\overrightarrow{OP} * \overrightarrow{OQ} * \overrightarrow{OR}$ (see Exercise 15). Specify also that A # P # B for any P on σ.
   (a) Let M be the point on σ such that $\overleftrightarrow{MO} \perp \overleftrightarrow{AB}$ (see Figure 3.48). Let $AMB = AM \cup MB$. For any point P on σ, prove that ray $\overrightarrow{OP}$ intersects AMB in a point P′ and that the mapping $P \mapsto P'$ is one-to-one from $\bar{\sigma}$ onto AMB.
   (b) Define P′ # Q′ # R′ to mean P # Q # R. If P′, Q′, and R′ all lie on segment AM or all lie on MB, prove that $P'' \# Q' \# R' \Rightarrow P' * Q' * R'$.

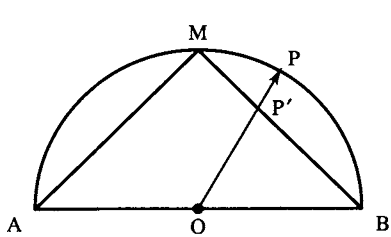


FIGURE 3.48

(c) Prove that Dedekind's axiom holds for AMB and hence for $\bar{\sigma}$ (use Exercise 7).

PROJECTS

1. Report on T. L. Heath's (1956) proof for the circular continuity principle.
2. Incidence, points, and lines in the real plane $\mathbb{R}^2$ were given in Major Exercise 9, Chapter 2. Distance is given by the usual Pythagorean formula
$$d(\mathrm{AB}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$
where $\mathrm{A} = (a_1, a_2)$, $\mathrm{B} = (b_1, b_2)$. Define A * B * C to mean $d(\mathrm{AC}) = d(\mathrm{AB}) + d(\mathrm{BC})$, and define AB ≅ CD to mean $d(\mathrm{AB}) = d(\mathrm{CD})$. Define ∢ABC ≅ ∢DEF if A, C, D, and F can be chosen on the sides of these angles so that AB ≅ ED, CB ≅ FE, and AC ≅ DF. With these interpretations, verify all the axioms for Euclidean geometry (see Moise, 1990, Chapter 26, or Borsuk and Szmielew, 1960, Chapter 4).
3. Suppose in Project 2 the field $\mathbb{R}$ of real numbers is replaced by an arbitrary *Euclidean field F* (an ordered field in which every positive number has a square root). Show that all the axioms for Euclidean geometry except Dedekind's and Archimedes' axioms are satisfied; show also that the circular continuity principle is satisfied.
4. In Euclidean geometry, Hilbert showed how to construct perpendiculars using ruler (marked straightedge) alone. His construction uses the theorem that the altitudes of a triangle are concurrent. Report on Hilbert's results. (Refer to D. Hilbert, 1987, p. 100.)