CHAPTER 4

NEUTRAL GEOMETRY

If only it could be proved . . . that "there is a Triangle whose angles are together not *less* than two right angles"! But alas, *that* is an *ignis fatuus* that has never yet been caught!

C. L. DODGSON (LEWIS CARROLL)

GEOMETRY WITHOUT THE PARALLELL AXIOM

In the exercises of the previous chapter you gained experience in proving some elementary results from Hilbert's axioms. Many of these results were taken for granted by Euclid. You can see that filling in the gaps and rigorously proving every detail is a long task. In any case, we must show that Euclid's postulates are consequences of Hilbert's. We have seen that Euclid's first postulate is the same as Hilbert's Axiom I-1. In our new language, Euclid's second postulate says the following: given segments AB and CD, there exists a point E such that $A * B * E$ and $CD \cong BE$. This follows immediately from Hilbert's Axiom C-1 applied to the ray emanating from B opposite to $\overrightarrow{BA}$ (see Figure 4.1).

The third postulate of Euclid becomes a definition in Hilbert's system. The *circle* with *center* O and *radius* OA is defined as the set of all points P such that OP is congruent to OA. Axiom C-1 then guarantees that on every ray emanating from O there exists such a point P.

The fourth postulate of Euclid—all right angles are congruent—becomes a theorem in Hilbert's system, as was shown in Proposition 3.23.

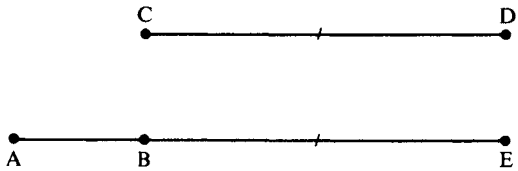

FIGURE 4.1

Euclid's parallel postulate is discussed later in this chapter. In this chapter we shall be interested in neutral geometry — by definition, all those geometric theorems that can be proved using only the axioms of incidence, betweenness, congruence, and continuity and without using the axiom of parallelism. Every result proved previously is a theorem in neutral geometry. You should review all the statements in the theorems, propositions, and exercises of Chapter 3 because they will be used throughout the book. Our proofs will be less formal henceforth.

What is the purpose of studying neutral geometry? We are not interested in studying it for its own sake. Rather, we are trying to clarify the role of the parallel postulate by seeing which theorems in the geometry do not depend on it, i.e., which theorems follow from the other axioms alone without ever using the parallel postulate in proofs. This will enable us to avoid many pitfalls and to see much more clearly the logical structure of our system. Certain questions that can be answered in Euclidean geometry (e.g., whether there is a unique parallel through a given point) may not be answerable in neutral geometry because its axioms do not give us enough information.

ALTERNATE INTERIOR ANGLE THEOREM

The next theorem requires a definition: let t be a transversal to lines l and l', with t meeting l at B and l' at B′. Choose points A and C on l such that A * B * C; choose points A′ and C′ on l' such that A and A′ are on the same side of t and such that A′ * B′ * C′. Then the following four angles are called *interior:* ∢A′B′B, ∢ABB′, ∢C′B′B, ∢CBB′. The two pairs (∢ABB′, ∢C′B′B) and (∢A′B′B, ∢CBB′) are called pairs of *alternate interior angles* (see Figure 4.2).

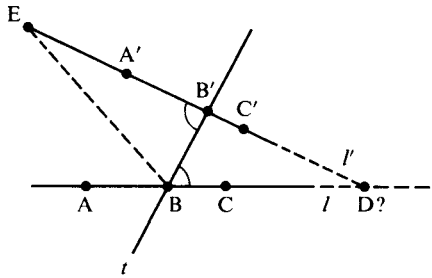

FIGURE 4.2

THEOREM 4.1 (Alternate Interior Angle Theorem). If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.

Proof:
Given $\measuredangle A'B'B \cong \measuredangle CBB'$. Assume on the contrary l and l' meet at a point D. Say D is on the same side of t as C and C′. There is a point E on $\overrightarrow{B'A'}$ such that $B'E \cong BD$ (Axiom C-1). Segment BB′ is congruent to itself, so that $\Delta B'BD \cong \Delta BB'E$ (SAS). In particular, $\measuredangle DB'B \cong \measuredangle EBB'$. Since $\measuredangle DB'B$ is the supplement of $\measuredangle EB'B$, $\measuredangle EBB'$ must be the supplement of $\measuredangle DBB'$ (Proposition 3.14 and Axiom C-4). This means that E lies on l, and hence l and l' have the two points E and D in common, which contradicts Proposition 2.1 of incidence geometry. Therefore, $l \parallel l'$. ■

This theorem has two very important corollaries.

COROLLARY 1. Two lines perpendicular to the same line are parallel. Hence, the perpendicular dropped from a point P not on line l to l is *unique* (and the point at which the perpendicular intersects l is called its *foot*).

Proof:
If l and l' are both perpendicular to t, the alternate interior angles are right angles and hence are congruent (Proposition 3.23). ■

COROLLARY 2. If l is any line and P is any point not on l, there exists at least one line m through P parallel to l (see Figure 4.3).

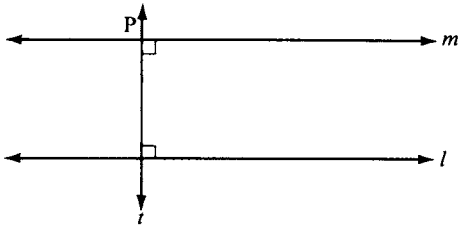

FIGURE 4.3

Proof:
There is a line t through P perpendicular to l, and again there is a unique line m through P perpendicular to t (Proposition 3.16). Since l and m are both perpendicular to t, Corollary 1 tells us that $l \parallel m$. (This construction will be used repeatedly.) ■

To repeat, there always exists a line m through P parallel to l—this has been proved in neutral geometry. But we don't know that m is *unique.* Although Hilbert's parallel postulate says that m is indeed unique, we are not assuming that postulate. We must keep our minds open to the strange possibility that there may be other lines through P parallel to l.

Warning. You are accustomed in Euclidean geometry to use the *converse* of Theorem 4.1, which states, "If two lines are parallel, then alternate interior angles cut by a transversal are congruent." We haven't proved this converse, so don't use it! (It turns out to be logically equivalent to the parallel postulate—see Exercise 5.)

EXTERIOR ANGLE THEOREM

Before we continue our list of theorems, we must first make another definition: an angle supplementary to an angle of a triangle is called an *exterior angle* of the triangle; the two angles of the triangle not adjacent to this exterior angle are called the *remote interior angles.* The following theorem is a consequence of Theorem 4.1:

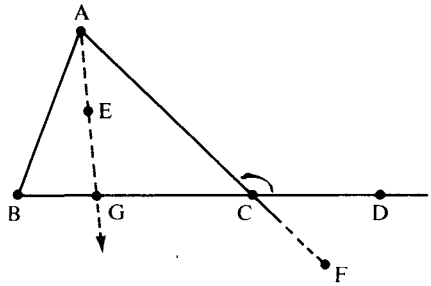

FIGURE 4.4

THEOREM 4.2 (Exterior Angle Theorem). An exterior angle of a triangle is greater than either remote interior angle (see Figure 4.4).

To prove $\measuredangle ACD$ is greater than $\measuredangle B$ and $\measuredangle A$:

Proof:
Consider the remote interior angle $\measuredangle BAC$. If $\measuredangle BAC \cong \measuredangle ACD$, then $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ (Theorem 4.1), which contradicts the hypothesis that these lines meet at B. Suppose $\measuredangle BAC$ were greater than $\measuredangle ACD$ (RAA hypothesis). Then there is a ray $\overrightarrow{AE}$ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ such that $\measuredangle ACD \cong \measuredangle CAE$ (by definition). This ray $\overrightarrow{AE}$ intersects BC in a point G (crossbar theorem, Chapter 3). But according to Theorem 4.1, lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{CD}$ are parallel. Thus, $\measuredangle BAC$ cannot be greater than $\measuredangle ACD$ (RAA conclusion). Since $\measuredangle BAC$ is also not congruent to $\measuredangle ACD$, $\measuredangle BAC$ must be less than $\measuredangle ACD$ (Proposition 3.21(a)).

For remote angle $\measuredangle ABC$, use the same argument applied to exterior angle $\measuredangle BCF$, which is congruent to $\measuredangle ACD$ by the vertical angle theorem (Proposition 3.15(a)). ■

The exterior angle theorem will play a very important role in what follows. It was the 16th proposition in Euclid's *Elements*. Euclid's proof had a gap due to reasoning from a diagram. He considered the line $\overleftrightarrow{BM}$ joining B to the midpoint of AC and he constructed point B′ such that $B * M * B'$ and $BM \cong MB'$ (Axiom C-1). He then assumed from the diagram that B′ lay in the interior of $\measuredangle ACD$ (see Figure 4.5). Since $\measuredangle B'CA \cong \measuredangle A$ (SAS), Euclid concluded correctly that $\measuredangle ACD > \measuredangle A$.

The gap in Euclid's argument can easily be filled with the tools we have developed. Since segment BB′ intersects AC at M, B and B′ are

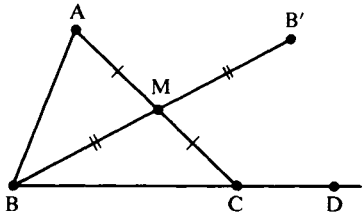

FIGURE 4.5

on opposite sides of $\overleftrightarrow{AC}$ (by definition). Since BD meets $\overleftrightarrow{AC}$ at C, B and D are also on opposite sides of $\overleftrightarrow{AC}$. Hence, B′ and D are on the same side of $\overleftrightarrow{AC}$ (Axiom B-4). Next, B′ and M are on the same side of $\overleftrightarrow{CD}$, since segment MB′ does not contain the point B at which $\overleftrightarrow{MB'}$ meets $\overleftrightarrow{CD}$ (by construction of B′ and Axioms B-1 and B-3). Also, A and M are on the same side of $\overleftrightarrow{CD}$ because segment AM does not contain the point C at which $\overleftrightarrow{AM}$ meets $\overleftrightarrow{CD}$ (by definition of midpoint and Axiom B-3). So again, Separation Axiom B-4 ensures that A and B′ are on the same side of $\overleftrightarrow{CD}$. By definition of "interior" (in Chapter 3, preceding Proposition 3.7), we have shown that B′ lies in the interior of $\sphericalangle ACD$.

Note on Elliptic Geometry. Figure 3.24 shows a triangle on the sphere with both an exterior angle and a remote interior angle that are right angles, so the exterior angle theorem doesn't hold. Our proof of it was based on the alternate interior angle theorem, which can't hold in elliptic geometry because there are no parallels. The proof we gave of Theorem 4.1 breaks down in elliptic geometry because Axiom B-4, which asserts that a line separates the plane into two sides, doesn't hold; we knew points E and D in that proof were distinct because they lay on opposite sides of line *t*. Or, thinking in terms of spherical geometry, where a great circle does separate the sphere into two hemispheres, if points E and D are distinct, there is no contradiction because great circles do meet in two antipodal points.

Euclid's proof of Theorem 4.2 breaks down on the sphere because "lines" are circles and if segment BM is long enough, the reflected point B′ might lie on it (e.g., if BM is a semicircle, B′ = B).

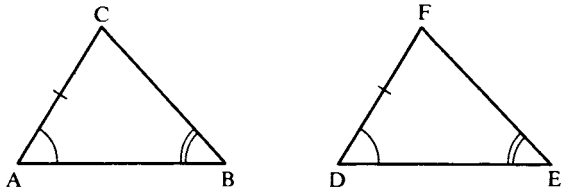

FIGURE 4.6

As a consequence of the exterior angle theorem (and our previous results), you can now prove as exercises the following familiar propositions.

PROPOSITION 4.1 (SAA Congruence Criterion). Given $AC \cong DF$, $\measuredangle A \cong \measuredangle D$, and $\measuredangle B \cong \measuredangle E$. Then $\triangle ABC \cong \triangle DEF$ (Figure 4.6).

PROPOSITION 4.2. Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other (Figure 4.7).

PROPOSITION 4.3 (Midpoints). Every segment has a unique midpoint.

PROPOSITION 4.4 (Bisectors). (a) Every angle has a unique bisector. (b) Every segment has a unique perpendicular bisector.

PROPOSITION 4.5. In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle, i.e., $AB > BC$ if and only if $\measuredangle C > \measuredangle A$.

PROPOSITION 4.6. Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\measuredangle B < \measuredangle B'$ if and only if $AC < A'C'$.

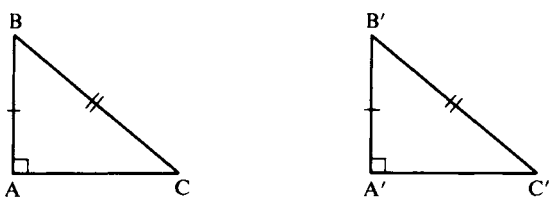

FIGURE 4.7

MEASURE OF ANGLES AND SEGMENTS

Thus far in our treatment of geometry we have refrained from using numbers that measure the sizes of angles and segments — this was in keeping with the spirit of Euclid. From now on, however, we will not be so austere. The next theorem (Theorem 4.3) asserts the possibility of measurement and lists its properties. The proof requires the axioms of continuity for the first time (in keeping with the elementary level of this book, the interested reader is referred to Borsuk and Szmielew, 1960, Chapter 3, Sections 9 and 10). In some popular treatments of geometry this theorem is taken as an axiom (ruler-and-protractor postulates — see Moise, 1990). The familiar notation $(\sphericalangle A)^\circ$ will be used for the number of degrees in $\sphericalangle A$, and the length of segment $\underline{AB}$ (with respect to some unit of measurement) will be denoted by $\overline{AB}$.

THEOREM 4.3. A. There is a unique way of assigning a degree measure to each angle such that the following properties hold (refer to Figure 4.8):

(0) $(\sphericalangle A)^\circ$ is a real number such that $0 < (\sphericalangle A)^\circ < 180^\circ$
(1) $(\sphericalangle A)^\circ = 90^\circ$ if and only if $\sphericalangle A$ is a right angle.
(2) $(\sphericalangle A)^\circ = (\sphericalangle B)^\circ$ if and only if $\sphericalangle A \cong \sphericalangle B$.
(3) If $\overrightarrow{AC}$ is interior to $\sphericalangle DAB$, then $(\sphericalangle DAB)^\circ = (\sphericalangle DAC)^\circ + (\sphericalangle CAB)^\circ$.
(4) For every real number x between 0 and 180, there exists an angle $\sphericalangle A$ such that $(\sphericalangle A)^\circ = x^\circ$.
(5) If $\sphericalangle B$ is supplementary to $\sphericalangle A$, then $(\sphericalangle A)^\circ + (\sphericalangle B)^\circ = 180^\circ$.
(6) $(\sphericalangle A)^\circ > (\sphericalangle B)^\circ$ if and only if $\sphericalangle A > \sphericalangle B$.

B. Given a segment OI, called a *unit segment*. Then there is a unique way of assigning a length $\overline{AB}$ to each segment AB such that the

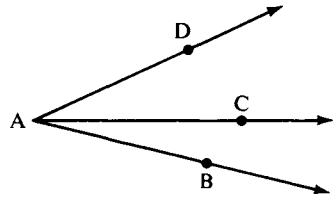

FIGURE 4.8

following properties hold:

(7) $\overline{AB}$ is a positive real number and $\overline{OI} = 1$.
(8) $\overline{AB} = \overline{CD}$ if and only if $\underline{AB} \cong \underline{CD}$.
(9) $\underline{A * B * C}$ if and only if $\overline{AC} = \overline{AB} + \overline{BC}$.
(10) $\overline{AB} < \overline{CD}$ if and only if $AB < CD$.
(11) For every positive real number x, there exists a segment AB such that $\overline{AB} = x$.

Note. So as not to mystify you, here is the method for assigning lengths. We start with a segment OI whose length will be 1. Then any segment obtained by laying off n copies of OI will have length n. By Archimedes' axiom, every other segment AB will have its endpoint B between two points B_{n-1} and B_n such that $\overline{AB_{n-1}} = n - 1$ and $\overline{AB_n} = n$; then $\overline{AB}$ will have to equal $\overline{AB_{n-1}} + \overline{B_{n-1}B}$ by condition (9) of Theorem 4.3, so we may assume $n = 1$ and $B_{n-1} = A$. If B is the midpoint $B_{1/2}$ of AB_1, we set $\overline{AB_{1/2}} = \frac{1}{2}$; otherwise B lies either in $AB_{1/2}$ or in $B_{1/2}B_1$, say, in $AB_{1/2}$. If then B is the midpoint $B_{1/4}$ of $AB_{1/2}$, we set $\overline{AB_{1/4}} = \frac{1}{4}$; otherwise B lies in $AB_{1/4}$, say, and we continue the process. Eventually B will either be obtained as the midpoint of some segment whose length has been determined, in which case $\overline{AB}$ will be determined to some dyadic rational number $a/2^n$; or the process will continue indefinitely, in which case $\overline{AB}$ will be the limit of an infinite sequence of dyadic rational numbers; i.e., $\overline{AB}$ will be determined as an infinite decimal with respect to the base 2.

The axioms of continuity are not needed if one merely wants to define addition for congruence classes of segments and then prove the triangle inequality (Corollary 2 to Theorem 4.3; see Borsuk and Szmielew, 1960, pp. 103–108, for a definition of this operation). It is in order to prove Theorem 4.4, Major Exercise 8, and the parallel projection theorem that we need the measurement of angles and segments by real numbers, and for such measurement Archimedes' axiom is required. However, parts 4 and 11 of Theorem 4.3, the proofs for which require Dedekind's axiom, are never used in proofs in this book. See Appendix B for coordinatization without continuity axioms.

Using degree notation, $\sphericalangle A$ is defined as *acute* if $(\sphericalangle A)^\circ < 90^\circ$, and *obtuse* if $(\sphericalangle A)^\circ > 90^\circ$. Combining Theorems 4.2 and 4.3 gives

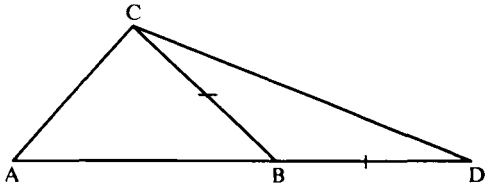

FIGURE 4.9

the following corollary, which is essential for proving the Saccheri-Legendre theorem.

COROLLARY 1. The sum of the degree measures of any *two* angles of a triangle is less than 180°.

The only immediate application of segment measurement that we will make is in the proof of the next corollary, the famous "triangle inequality."

COROLLARY 2 (Triangle Inequality). If A, B, and C are three noncollinear points, then $\overline{AC} < \overline{AB} + \overline{BC}$.

Proof:

(1) There is a unique point D such that $A * B * D$ and $BD \cong BC$ (Axiom C-1 applied to the ray opposite to $\overrightarrow{BA}$). (See Figure 4.9.)
(2) Then $\measuredangle BCD \cong \measuredangle BDC$ (Proposition 3.10: base angles of an isosceles triangle).
(3) $\overline{AD} = \overline{AB} + \overline{BD}$ (Theorem 4.3(9)) and $\overline{BD} = \overline{BC}$ (step 1 and Theorem 4.3(8)); substituting gives $\overline{AD} = \overline{AB} + \overline{BC}$.
(4) $\overrightarrow{CB}$ is between $\overrightarrow{CA}$ and $\overrightarrow{CD}$ (Proposition 3.7); hence, $\measuredangle ACD > \measuredangle BCD$ (by definition).
(5) $\measuredangle ACD > \measuredangle ADC$ (steps 2 and 4; Proposition 3.21(c)).
(6) $AD > AC$ (Proposition 4.5).
(7) Hence, $\overline{AB} + \overline{BC} > \overline{AC}$ (Theorem 4.3(10); steps 3 and 6). ■

SACCHERI-LEGENDRE THEOREM

The following very important theorem also requires an axiom of continuity (Archimedes' axiom) for its proof.

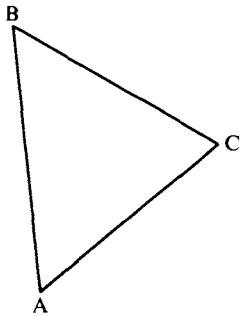

FIGURE 4.10 $(\measuredangle A)^\circ + (\measuredangle B)^\circ + (\measuredangle C)^\circ \leqq 180^\circ$.

THEOREM 4.4 (Saccheri-Legendre). The sum of the degree measures of the three angles in any triangle is *less than or equal to* 180°.

This result may strike you as peculiar, since you are accustomed to the notion of an *exact* sum of 180°. Nevertheless, this exactness cannot be proved in neutral geometry! Saccheri tried, but the best he could conclude was "less than or equal." Max Dehn showed in 1900 that there is no way to prove this theorem without Archimedes' axiom.[1] The idea of the proof is as follows:

Assume, on the contrary, that the angle sum of $\triangle ABC$ is greater than 180°, say $180^\circ + p^\circ$, where p is a positive number. It is possible (by a trick you will find in Exercise 15) to replace $\triangle ABC$ with another triangle that has the same angle sum as $\triangle ABC$ but in which one angle has at most half the number of degrees as $(\measuredangle A)^\circ$ We can repeat this trick to get another triangle that has the same angle sum $180^\circ + p^\circ$ but in which one angle has at most one-fourth the number of degrees as $(\measuredangle A)^\circ$. The Archimedean property of real numbers guarantees that if we repeat this construction enough times, we will eventually obtain a triangle that has angle sum $180^\circ + p^\circ$ but in which one angle has degree measure at most p°. The sum of the degree measures of the other *two* angles will be greater than or equal to 180°, contradicting Corollary 1 to Theorem 4.3. This proves the theorem.

You should prove the following consequence of the Saccheri-Legendre theorem as an exercise.

[1] See the heuristic argument in Project 1. The full significance of Archimedes' axiom was first grasped in the 1880s by M. Pasch and O. Stolz. G. Veronese and T. Levi-Civita developed the first non-Archimedean geometry. Also see Appendix B.

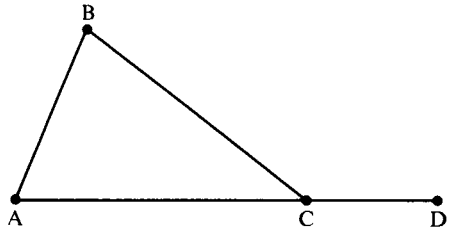

FIGURE 4.11 $(\measuredangle A)^\circ + (\measuredangle B)^\circ \leqq (\measuredangle BCD)^\circ$.

COROLLARY 1. The sum of the degree measures of two angles in a triangle is less than or equal to the degree measure of their remote exterior angle (see Figure 4.11).

It is natural to generalize the Saccheri-Legendre theorem to polygons other than triangles. For example, let us prove that the angle sum of a quadrilateral ABCD is at most 360°. Break □ABCD into two triangles, △ABC and △ADC, by the diagonal AC (see Figure 4.12). By the Saccheri-Legendre theorem,

$$(\measuredangle B)^\circ + (\measuredangle BAC)^\circ + (\measuredangle ACB)^\circ \leqq 180^\circ$$

and

$$(\measuredangle D)^\circ + (\measuredangle DAC)^\circ + (\measuredangle ACD)^\circ \leqq 180^\circ.$$

Theorem 4.3(3) gives us the equations

$$(\measuredangle BAC)^\circ + (\measuredangle DAC)^\circ = (\measuredangle BAD)^\circ$$

and

$$(\measuredangle ACB)^\circ + (\measuredangle ACD)^\circ = (\measuredangle BCD)^\circ$$

Using these equations, we add the two inequalities to obtain the desired inequality

$$(\measuredangle B)^\circ + (\measuredangle D)^\circ + (\measuredangle BAD)^\circ + (\measuredangle BCD)^\circ \leqq 360^\circ$$

Unfortunately, there is a gap in this simple argument! To get the equations used above, we assumed by looking at the diagram (Figure

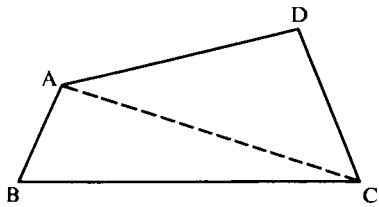

FIGURE 4.12

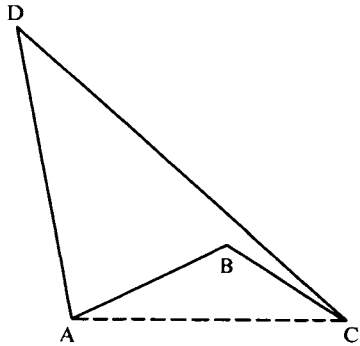

FIGURE 4.13

4.12) that C was interior to ∢BAD and that A was interior to ∢BCD. But what if the quadrilateral looked like Figure 4.13? In this case the equations would not hold. To prevent such a case, we must add a hypothesis; we must assume that the quadrilateral is "convex."

DEFINITION. Quadrilateral □ABCD is called *convex* if it has a pair of opposite sides, e.g., AB and CD, such that CD is contained in one of the half-planes bounded by $\overleftrightarrow{AB}$ and AB is contained in one of the half-planes bounded by $\overleftrightarrow{CD}$.[2]

The assumption made above is now justified by starting with a convex quadrilateral. Thus, we have proved the following corollary:

COROLLARY 2. The sum of the degree measures of the angles in any *convex* quadrilateral is at most 360°.

Note. The Saccheri-Legendre theorem is false in elliptic geometry (see Figure 3.24.). In fact, it can be proved in elliptic geometry that the angle sum of a triangle is always greater than 180° (see Kay, 1969). Since a triangle can have two or three right angles, a *hypotenuse,*

[2] It can be proved that this condition also holds for the other pair of opposite sides, AD and BC—see Exercise 23 in this chapter. The use of the word "convex" in this definition does not agree with its use in Exercise 19, Chapter 3; a convex quadrilateral is obviously not a "convex set" as defined in that exercise. However, we can define the *interior* of a convex quadrilateral □ABCD as follows: each side of □ABCD determines a half-plane containing the opposite side; the interior of □ABCD is then the intersection of the four half-planes so determined. You can then prove that the interior of a convex quadrilateral is a convex set (which is one of the problems in Exercise 25).

defined as a side opposite a right angle, need not be unique, and a *leg*, defined as a side of a right triangle not opposite a right angle, need not exist (and if opposite an obtuse angle, a leg could be longer than a hypotenuse).

EQUIVALENCE OF PARALLEL POSTULATES

We shall now prove the equivalence of Euclid's fifth postulate and Hilbert's parallel postulate. Note, however, that we are not proving either or both of the postulates; we are only proving that we *can* prove one *if* we first assume the other. We shall first state Euclid V (all the terms in the statement have now been defined carefully).

EUCLID'S POSTULATE V. If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

THEOREM 4.5. Euclid's fifth postulate $\Leftrightarrow$ Hilbert's parallel postulate.

Proof:
First, assume Hilbert's postulate. The situation of Euclid V is shown in Figure 4.14. $(\measuredangle 1)° + (\measuredangle 2)° < 180°$ (hypothesis) and $(\measuredangle 1)° + (\measuredangle 3)° = 180°$ (supplementary angles, Theorem 4.3(5)). Hence, $(\measuredangle 2)° < 180° - (\measuredangle 1)° = (\measuredangle 3)°$. There is a unique ray $\overrightarrow{B'C'}$ such that $\measuredangle 3$ and $\measuredangle C'B'B$ are congruent alternate interior angles (Axiom C-4). By Theorem 4.1, $\overleftrightarrow{B'C'}$ is parallel to l. Since

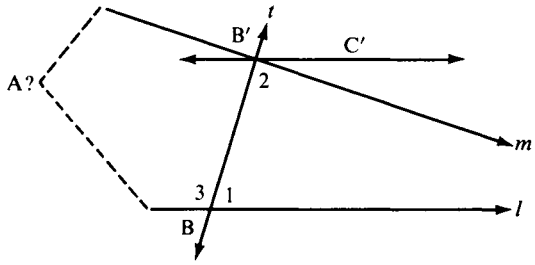

FIGURE 4.14

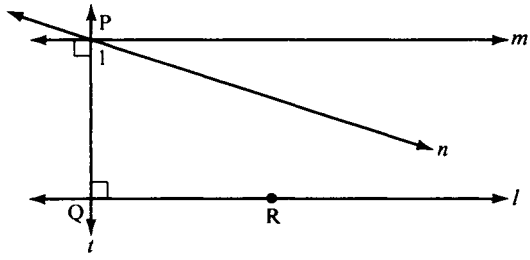

FIGURE 4.15

$m \neq \overleftrightarrow{B'C'}$, m meets l (Hilbert's postulate). To conclude, we must prove that m meets l on the same side of t as C′. Assume, on the contrary, that they meet at a point A on the opposite side. Then $\measuredangle 2$ is an exterior angle of $\triangle ABB'$. Yet it is smaller than the remote interior $\measuredangle 3$. This contradiction of Theorem 4.2 proves Euclid V (RAA).

Conversely, assume Euclid V and refer to Figure 4.15, the situation of Hilbert's postulate. Let t be the perpendicular to l through P, and m the perpendicular to t through P. We know that $m \parallel l$ (Corollary 1 to Theorem 4.1). Let n be any other line through P. We must show that n meets l. Let $\measuredangle 1$ be the acute angle n makes with t (which angle exists because $n \neq m$). Then $(\measuredangle 1)^\circ + (\measuredangle PQR)^\circ < 90^\circ + 90^\circ = 180^\circ$. Thus, the hypothesis of Euclid V is satisfied. Hence, n meets l, proving Hilbert's postulate. ■

Since Hilbert's parallel postulate and Euclid V are logically equivalent in the context of neutral geometry, Theorem 4.5 allows us to use them interchangeably. You will prove as exercises that the following statements are also logically equivalent to the parallel postulate.[3]

PROPOSITION 4.7. Hilbert's parallel postulate $\Leftrightarrow$ if a line intersects one of two parallel lines, then it also intersects the other.

PROPOSITION 4.8. Hilbert's parallel postulate $\Leftrightarrow$ converse to Theorem 4.1 (alternate interior angles).

PROPOSITION 4.9. Hilbert's parallel postulate $\Leftrightarrow$ if t is a transversal to l and m, $l \parallel m$, and $t \perp l$, then $t \perp m$.

[3] Transitivity of parallelism is also logically equivalent to the parallel postulate.

PROPOSITION 4.10. Hilbert's parallel postulate $\Leftrightarrow$ if $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n$.

The next proposition is another statement logically equivalent to Hilbert's parallel postulate, but at this point we can only prove the implication in one direction (the other implication is proved in Chapter 5; see Exercise 14).

PROPOSITION 4.11. Hilbert's parallel postulate $\Rightarrow$ the angle sum of every triangle is 180°.

ANGLE SUM OF A TRIANGLE

We define the *angle sum* of triangle $\triangle ABC$ as $(\sphericalangle A)° + (\sphericalangle B)° + (\sphericalangle C)°$, which is a certain number of degrees $\leqq 180°$ (by the Saccheri-Legendre theorem). We define the *defect* δABC to be 180° minus the angle sum. In Euclidean geometry we are accustomed to having no "defective" triangles, i.e., we are accustomed to having the defect equal zero (Proposition 4.11).

The main purpose of this section is to show that if *one* defective triangle exists, then *all* triangles are defective. Or, put in the contrapositive form, if one triangle has angle sum 180°, then so do all others. We are not asserting that one such triangle does exist, nor are we asserting the contrary; we are only examining the hypothesis that one might exist.

THEOREM 4.6. Let $\triangle ABC$ be any triangle and D a point between A and B (Figure 4.16). Then $\delta ABC = \delta ACD + \delta BCD$ *(additivity of the defect)*.

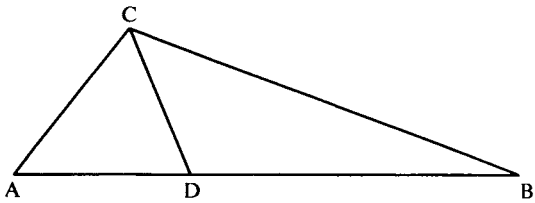

FIGURE 4.16

Proof:
Since $\overrightarrow{CD}$ is interior to $\sphericalangle ACB$ (Proposition 3.7), $(\sphericalangle ACB)^\circ = (\sphericalangle ACD)^\circ + (\sphericalangle BCD)^\circ$ (by Theorem 4.3(3)). Since $\sphericalangle ADC$ and $\sphericalangle BDC$ are supplementary angles, $180^\circ = (\sphericalangle ADC)^\circ + (\sphericalangle BDC)^\circ$ (by Theorem 4.3(5)). To obtain the additivity of the defect, all we have to do is write down the definition of the defect (180° minus the angle sum) for each of the three triangles under consideration and substitute the two equations above (Exercise 1). ■

COROLLARY. Under the same hypothesis, the angle sum of $\triangle ABC$ is equal to 180° if and only if the angle sums of both $\triangle ACD$ and $\triangle BCD$ are equal to 180°.

Proof:
If $\triangle ACD$ and $\triangle BCD$ both have defect zero, then defect of $\triangle ABC = 0 + 0 = 0$ (Theorem 4.6). Conversely, if $\triangle ABC$ has defect zero, then, by Theorem 4.6, $\delta ACD + \delta BCD = 0$. But the defect of a triangle can never be negative (Saccheri-Legendre theorem). Hence, $\triangle ACD$ and $\triangle BCD$ each have defect zero (the sum of two nonnegative numbers equals zero only when each equals zero). ■

Next, recall that by definition a *rectangle* is a quadrilateral whose four angles are right angles. Hence, the angle sum of a rectangle is 360°. Of course, we don't yet know whether rectangles exist in neutral geometry. (Try to construct one without using the parallel postulate or any statement logically equivalent to it—see Exercise 19.)

The next theorem is the result we seek. Its proof will be given in five steps.

THEOREM 4.7. If a triangle exists whose angle sum is 180°, then a rectangle exists. If a rectangle exists, then every triangle has angle sum equal to 180°.

Proof:
(1) Construct a *right* triangle having angle sum 180°.

Let $\triangle ABC$ be the given triangle with defect zero (hypothesis). Assume it is not a right triangle; otherwise we are done. At least two of the angles in this triangle are acute, since the angle

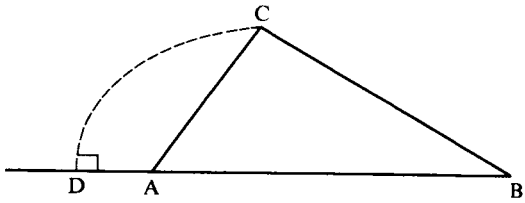

FIGURE 4.17

sum of two angles in a triangle must be less than 180° (corollary to Theorem 4.3); e.g., assume ∢A and ∢B are acute. Let CD be the altitude from vertex C (which exists, by Proposition 3.16). We claim that D lies between A and B. Assume the contrary, that D * A * B (see Figure 4.17). Then remote interior angle ∢CDA is greater than exterior angle ∢CAB, contradicting Theorem 4.2. Similarly, if A * B * D, we get a contradiction. Thus, A * D * B (Axiom B-3); see Figure 4.18. It now follows from the corollary to Theorem 4.6 that each of the right triangles ΔADC and ΔBDC has defect zero.

(2) From a right triangle of defect zero construct a rectangle.

Let ΔCDB be a right triangle of defect zero with ∢D a right angle. By Axiom C-4, there is a unique ray $\overrightarrow{CX}$ on the opposite side of $\overleftrightarrow{CB}$ from D such that ∢DBC ≅ ∢BCX. By Axiom C-1, there is a unique point E on $\overrightarrow{CX}$ such that CE ≅ BD (Figure 4.19).Then ΔCDB ≅ ΔBEC (SAS). Hence, ΔBEC is also a right triangle of defect zero with right angle at E. Also, since (∢DBC)° + (∢BCD)° = 90° by our hypothesis, we obtain by substitution (∢ECB)° + (∢BCD)° = 90° and (∢DBC)° + (∢EBC)° = 90° Moreover, B is an interior point of ∢ECD, since the alternate interior angle theorem implies $\overleftrightarrow{CE} \parallel \overleftrightarrow{DB}$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{BE}$ and C is interior to ∢EBD (for the same reason).

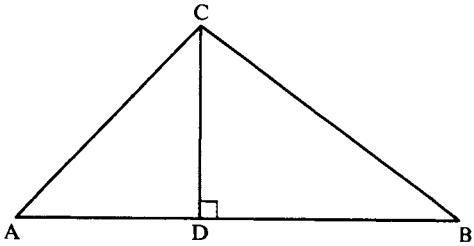

FIGURE 4.18

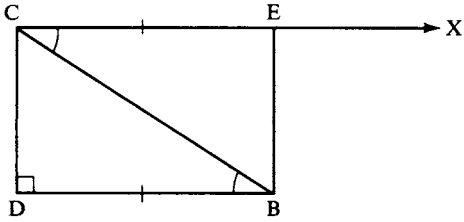

FIGURE 4.19

Thus, we can apply Theorem 4.3(3) to conclude that $(\measuredangle ECD)^\circ = 90^\circ = (\measuredangle EBD)^\circ$. This proves that $\square CDBE$ is a rectangle.

(3) From one rectangle, construct "arbitrarily large" rectangles.

More precisely, given any right triangle $\Delta D'E'C'$, construct a rectangle $\square AFBC$ such that $AC > D'C'$ and $BC > E'C'$. This can be done using Archimedes' axiom. We simply "lay off" enough copies of the rectangle we have to achieve the result (see Figures 4.20 and 4.21; you can make this "laying off" precise as an exercise).

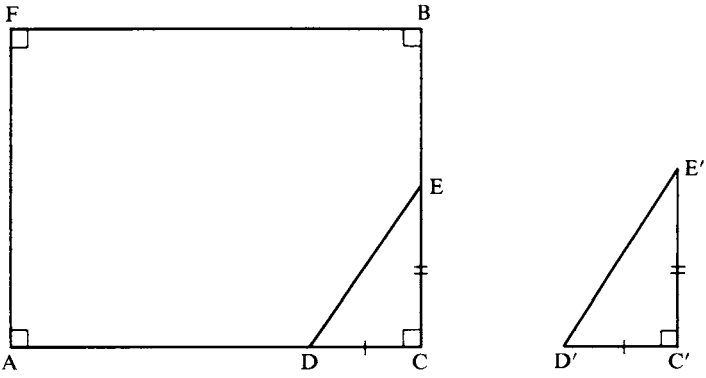

FIGURE 4.20

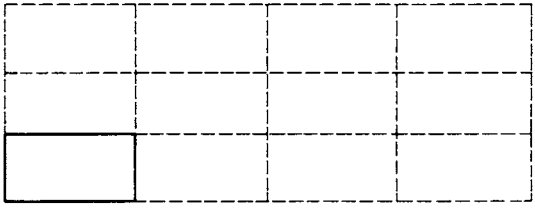

FIGURE 4.21

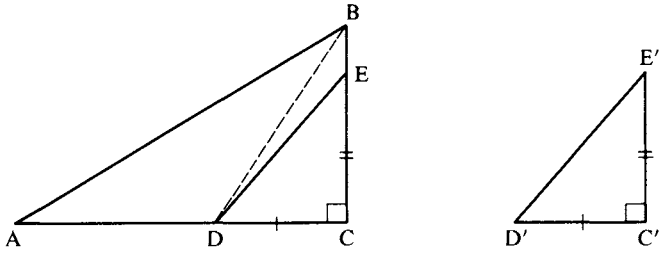

FIGURE 4.22

(4) Prove that all *right* triangles have defect zero.
This is achieved by "embedding" an arbitrary right triangle $\Delta D'C'E'$ in a rectangle, as in step 3, and then showing successively (by twice applying the corollary to Theorem 4.6) that ΔACB, ΔDCB, and ΔDCE each have defect zero (see Figure 4.22).

(5) If every *right* triangle has defect zero, then *every* triangle has defect zero.

As in step 1, drop an altitude to decompose an arbitrary triangle into two right triangles (Figure 4.18) and apply the corollary to Theorem 4.6. ■

Historians credit Theorem 4.7 to Saccheri and Legendre, but we will not name it after them, so as to avoid confusion with Theorem 4.4.

COROLLARY. If there exists a triangle with positive defect, then all triangles have positive defect.

REVIEW EXERCISE

Which of the following statements are correct?

(1) If two triangles have the same defect, they are congruent.
(2) Euclid's fourth postulate is a theorem in neutral geometry.

(3) Theorem 4.5 shows that Euclid's fifth postulate is a theorem in neutral geometry.
(4) The Saccheri-Legendre theorem tells us that some triangles exist that have angle sum less than 180° and some triangles exist that have angle sum equal to 180°.
(5) The alternative interior angle theorem states that if parallel lines are cut by a transversal, then alternate interior angles are congruent to each other.
(6) It is impossible to prove in neutral geometry that quadrilaterals exist.
(7) The Saccheri-Legendre theorem is false in Euclidean geometry because in Euclidean geometry the angle sum of any triangle is never less than 180°.
(8) According to our definition of "angle," the degree measure of an angle cannot equal 180°.
(9) The notion of one ray being "between" two others is undefined.
(10) It is impossible to prove in neutral geometry that parallel lines exist.
(11) The definition of "remote interior angle" given on p. 118 is incomplete because it used the word "adjacent," which has never been defined.
(12) An exterior angle of a triangle is any angle that is not in the interior of the triangle.
(13) The SSS criterion for congruence of triangles is a theorem in neutral geometry.
(14) The alternate interior angle theorem implies, as a special case, that if a transversal is perpendicular to one of two parallel lines, then it is also perpendicular to the other.
(15) Another way of stating the Saccheri-Legendre theorem is to say that the defect of a triangle cannot be negative.
(16) The ASA criterion for congruence of triangles is one of the axioms for neutral geometry.
(17) The proof of Theorem 4.7 depends on Archimedes' axiom.
(18) If $\triangle ABC$ is any triangle and C is any of its vertices, and if a perpendicular is dropped from C to $\overleftrightarrow{AB}$, then that perpendicular will intersect $\overleftrightarrow{AB}$ in a point between A and B.
(19) It is a theorem in neutral geometry that given any point P and any line l, there is at most one line through P perpendicular to l.
(20) It is a theorem in neutral geometry that vertical angles are congruent to each other.
(21) The proof of Theorem 4.2 (on exterior angles) uses Theorem 4.1 (on alternate interior angles).
(22) The gap in Euclid's attempt to prove Theorem 4.2 can be filled using our axioms of betweenness.

EXERCISES

The following are exercises in neutral geometry, unless otherwise stated. This means that in your proofs you are allowed to use only those results that have been given previously (including results from previous exercises). You are not allowed to use the parallel postulate or other results from Euclidean geometry that depend on it.

1. (a) Finish the last step in the proof of Theorem 4.6. (b) Prove that congruent triangles have the same defect. (c) Prove the corollary to Theorem 4.7. (d) Prove Corollary 1 to Theorem 4.3.
2. The Pythagorean theorem cannot be proved in neutral geometry (as you will show in Exercise 11(d), Chapter 6). Explain why the Euclidean proof suggested by Figure 1.15 of Chapter 1 is not valid in neutral geometry.
3. State the converse to Euclid's fifth postulate. Prove this converse as a theorem in neutral geometry.
4. Prove Proposition 4.7. Deduce as a corollary that transitivity of parallelism is equivalent to Hilbert's parallel postulate.
5. Prove Proposition 4.8. (Hint: Assume the converse to Theorem 4.1. Let m be the parallel to l through P constructed in the proof of Corollary 2 to Theorem 4.1 and let n be any parallel to l through P. Use the congruence of alternate interior angles and the uniqueness of perpendiculars to prove $m = n$. Assuming next the parallel postulate, use Axiom C-4 and an RAA argument to establish the converse to Theorem 4.1.)
6. Prove Proposition 4.9.
7. Prove Proposition 4.10.
8. Prove Proposition 4.11. (Hint: See Figure 4.23.)
9. The following purports to be a proof in neutral geometry of the SAA criterion for congruence. Find the flaw (see Figure 4.6).

   Given $AC \cong DF$, $\measuredangle A \cong \measuredangle D$, $\measuredangle B \cong \measuredangle E$. Then $\measuredangle C \cong \measuredangle F$, since $(\measuredangle C)^\circ = 180^\circ - (\measuredangle A)^\circ - (\measuredangle B)^\circ = 180^\circ - (\measuredangle D)^\circ - (\measuredangle E)^\circ = (\measuredangle F)^\circ$

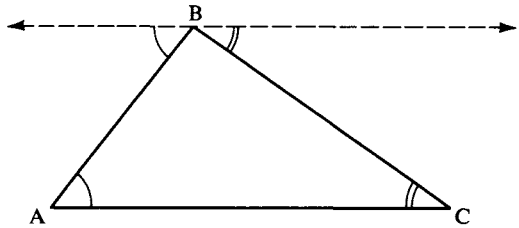

FIGURE 4.23

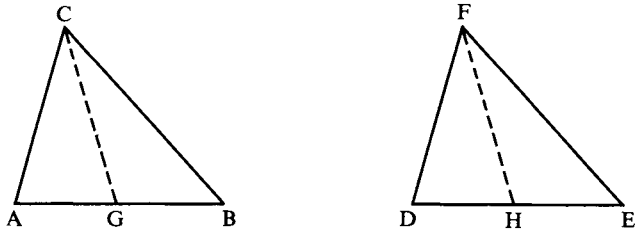

FIGURE 4.24

(Theorem 4.3(2)). Hence, $\triangle ABC \cong \triangle DEF$ by ASA (Proposition 3.17).

10. Here is a correct proof of the SAA criterion. Justify each step. (1) Assume side AB is not congruent to side DE. (2) Then $AB < DE$ or $DE < AB$. (3) If $DE < AB$, then there is a point G between A and B such that $AG \cong DE$ (see Figure 4.24). (4) Then $\triangle CAG \cong \triangle FDE$. (5) Hence, $\measuredangle AGC \cong \measuredangle E$. (6) It follows that $\measuredangle AGC \cong \measuredangle B$. (7) This contradicts a certain theorem (which ?). (8) Therefore, DE is not les than AB. (9) By a similar argument involving a point H between D and E, AB is not less than DE. (10) Hence, $AB \cong DE$. (11) Therefore, $\triangle ABC \cong \triangle DEF$.
11. Prove Proposition 4.2. (Hint: See Figure 4.7. On the ray opposite to $\overrightarrow{AC}$, lay off segment AD congruent to $A'C'$. First prove $\triangle DAB \cong \triangle C'A'B'$, and then use isosceles triangles and the SAA criterion to conclude.)
12. Here is a proof that segment AB has a midpoint. Justify each step (see Figure 4.25).
(1) Let C be any point not on $\overleftrightarrow{AB}$. (2) There is a unique ray $\overrightarrow{BX}$ on the opposite side of $\overleftrightarrow{AB}$ from C such that $\measuredangle CAB \cong \measuredangle ABX$. (3) There is a

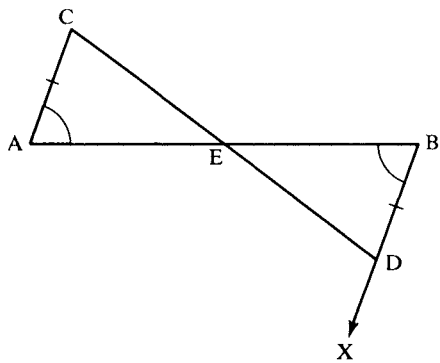

FIGURE 4.25

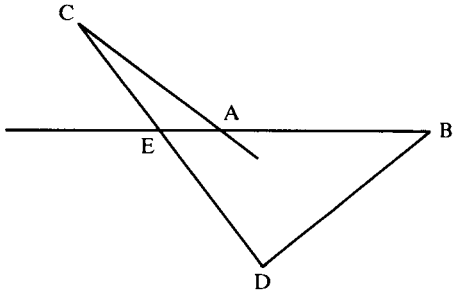

FIGURE 4.26

unique point D on $\overrightarrow{BX}$ such that AC ≅ BD. (4) D is on the opposite side of $\overleftrightarrow{AB}$ from C. (5) Let E be the point at which segment CD intersects $\overleftrightarrow{AB}$. (6) Assume E is not between A and B. (7) Then either E = A, or E = B, or E * A * B, or A * B * E. (8) $\overleftrightarrow{AC}$ is parallel to $\overleftrightarrow{BD}$. (9) Hence, E ≠ A and E ≠ B. (10) Assume E * A * B (Figure 4.26). (11) Since $\overleftrightarrow{CA}$ intersects side EB of △EBD at a point between E and B, it must also intersect either ED or BD. (12) Yet this is impossible. (13) Hence, A is not between E and B. (14) Similarly, B is not between A and E. (15) Thus, A * E * B (see Figure 4.25). (16) Then ∢AEC ≅ ∢BED. (17) △EAC ≅ △EBD. (18) Therefore, E is a midpoint of AB.

13. (a) Prove that segment AB has only one midpoint. (Hint: Assume the contrary and use Propositions 3.3 and 3.13 to derive a contradiction, or else put another possible midpoint E′ into Figure 4.25 and derive a contradiction from congruent triangles.)
    (b) Prove Proposition 4.4 on bisectors. (Hint: Use midpoints.)
14. Prove Corollary 1 to the Saccheri-Legendre theorem.
15. Prove the following result, needed to demonstrate the Saccheri-Legendre theorem (see Figure 4.27). Let D be the midpoint of BC and E the unique point on $\overleftrightarrow{AD}$ such that A * D * E and AD ≅ DE. Then △AEC has the same angle sum as △ABC, and either (∢EAC)° or (∢AEC)° is $\leqq \frac{1}{2}$ (∢BAC)°. (Hint: First show that △BDA ≅ △CDE, then that (∢EAC)° + (∢AEC)° = (∢BAC)°.)

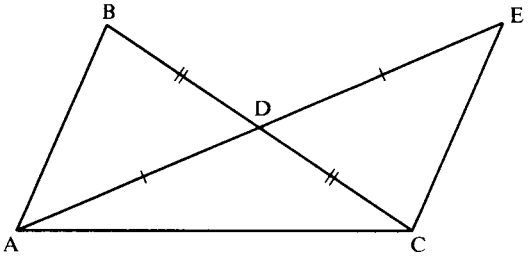

FIGURE 4.27

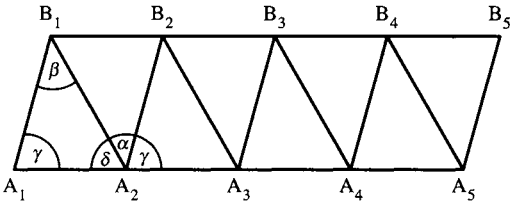

FIGURE 4.28

16. Here is another proof of Theorem 4.4 due to Legendre. Justify the unjustified steps: (1) Let $A_1A_2B_1$ be the given triangle, lay off n copies of segment A_1A_2, and construct a row of triangles $A_jA_{j+1}B_j$, $j = 1, \ldots, n$, congruent to $A_1A_2B_1$ as shown in Figure 4.28. (2) The $B_jA_{j+1}B_{j+1}$, $j = 1, \ldots, n$, are also congruent triangles, the last by construction of B_{n+1}. (3) With angles labeled as in Figure 4.28, $\alpha + \gamma + \delta = 180°$ and $\beta + \gamma + \delta$ equals the angle sum of $A_1A_2B_1$. (4) Assume on the contrary that $\beta > \alpha$. (5) Then $\overline{A_1A_2} > \overline{B_1B_2}$, by Proposition 4.6. (6) Also $\overline{A_1B_1} + n \cdot \overline{B_1B_2} + \overline{B_{n+1}A_{n+1}} > n \cdot \overline{A_1A_2}$, by repeated application of the triangle inequality. (7) $\overline{A_1B_1} \cong \overline{B_{n+1}A_{n+1}}$. (8) $2\overline{A_1B_1} > n(\overline{A_1A_2} - \overline{B_1B_2})$. (9) Since n was arbitrary, this contradicts Archimedes' axiom. (10) Hence the triangle has angle sum $\leqq 180°$.
17. Prove the following theorems:
    (a) Let γ be a circle with center O, and let A and B be two points on γ. The segment $\overline{AB}$ is called a *chord* of γ; let M be its midpoint. If $O \neq M$, then $\overleftrightarrow{OM}$ is perpendicular to $\overleftrightarrow{AB}$. (Hint: Corresponding angles of congruent triangles are congruent.)
    (b) Let AB be a chord of the circle γ having center O. Prove that the perpendicular bisector of AB passes through the center O of γ.
18. Prove the theorem of Thales *in Euclidean geometry* that an angle inscribed in a semicircle is a right angle. Prove in *neutral geometry* that this statement implies the existence of a right triangle with zero defect (see Figure 4.29).

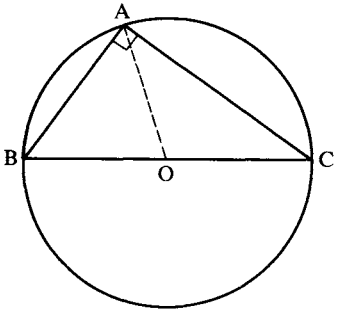

FIGURE 4.29

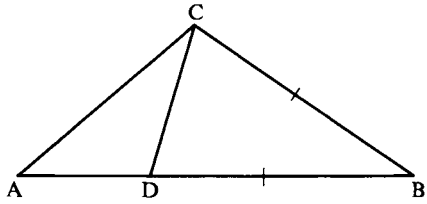

FIGURE 4.30

19. Find the flaw in the following argument purporting to construct a rectangle. Let A and B be any two points. There is a line l through A perpendicular to $\overleftrightarrow{AB}$ (Proposition 3.16) and, similarly, there is a line m through B perpendicular to $\overleftrightarrow{AB}$. Take any point C on m other than B. There is a line through C perpendicular to l—let it intersect l at D. Then □ABCD is a rectangle.
20. The sphere, with "lines" interpreted as great circles, is not a model of neutral geometry. Here is a proposed construction of a rectangle on a sphere. Let α, β be two circles of longitude and let them intersect the equator at A and D. Let γ be a circle of latitude in the northern hemisphere, intersecting α and β at two other points, B and C. Since circles of latitude are perpendicular to circles of longitude, the quadrilateral with vertices ABCD and sides the arcs of α, γ, and β and the equator traversed in going from A north to B east to C south to D west to A should be a rectangle. Explain why this construction doesn't work.
21. Prove Proposition 4.5. (Hint: If AB > BC, then let D be the point between A and B such that BD ≅ BC (Figure 4.30). Use isosceles triangle △CBD and exterior angle ∢BDC to show that ∢ACB > ∢A. Use this result and trichotomy of ordering to prove the converse.)
22. Prove Proposition 4.6. (Hint: Given ∢B < ∢B′. Use the hypothesis of Proposition 4.6 to reduce to the case A = A′, B = B′, and C interior to ∢ABC′, so that you must show AC < AC′ (see Figure 4.31). This is easy in case C = D, where point D is obtained from the crossbar theorem. In case C ≠ D, Proposition 4.5 reduces the problem to showing that ∢AC′C < ∢ACC′. In case B * D * C (as in Figure 4.31), you

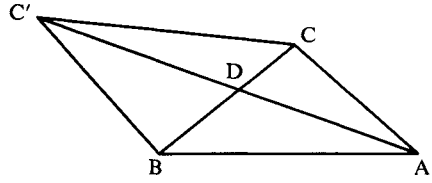

FIGURE 4.31 BC ≅ BC′.

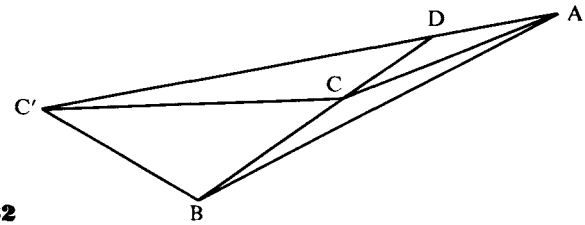

FIGURE 4.32

can prove this inequality using the congruence $\measuredangle BCC' \cong \measuredangle BC'C$. In case $B * C * D$ (Figure 4.32), apply the congruence $\measuredangle BCC' \cong \measuredangle BC'C$ and Theorem 4.2 to exterior angle $\measuredangle BCC'$ of $\Delta DCC'$ and exterior angle $\measuredangle DCC'$ of $\Delta BCC'$. (The converse implication in Proposition 4.6 follows from the direct implication, just shown, if you apply trichotomy.)

23. For the purpose of this exercise, call segments AB and CD *semiparallel* if segment AB does not intersect line $\overleftrightarrow{CD}$ and segment CD does not intersect line $\overleftrightarrow{AB}$. Obviously, if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then AB and CD are semiparallel, but the converse need not hold (see Figure 4.33). We have defined a quadrilateral to be convex if one pair of opposite sides is semiparallel. Prove that the other pair of opposite sides is also semiparallel. (Hint: Suppose AB is semiparallel to CD and assume, on the contrary, that AD meets $\overleftrightarrow{BC}$ in a point E. Use the definition of quadrilateral (Exercise 3, Chapter 1) to show either that $E * B * C$ or $B * C * E$; in either case, use Pasch's theorem to derive a contradiction.)
24. Prove that the diagonals of a convex quadrilateral intersect. (Hint: Apply the crossbar theorem.)
25. Prove that the intersection of convex sets (defined in Exercise 19, Chapter 3) is again a convex set. Use this result to prove that the interior of a convex quadrilateral is a convex set and that the point at which the diagonals intersect lies in the interior.
26. The *convex* hull of a set of points S is the intersection of all the convex sets containing S; i.e., it is the smallest convex set containing S. Prove that the convex hull of three noncollinear points A, B, and C consists of the sides and interior of ΔABC.

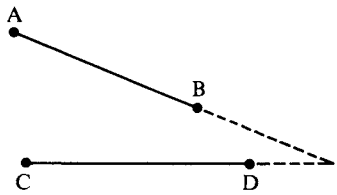

FIGURE 4.33

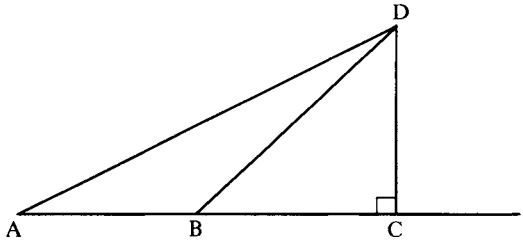

FIGURE 4.34

27. Given A * B * C and $\overleftrightarrow{DC} \perp \overleftrightarrow{AC}$. Prove that AD > BD > CD (Figure 4.34; use Proposition 4.5).
28. Given any triangle ΔDAC and any point B between A and C. Prove that either DB < DA or DB < DC. (Hint: Drop a perpendicular from D to $\overleftrightarrow{AC}$ and use the previous exercise.)
29. Prove that the interior of a circle is a convex set. (Hint: Use the previous exercise.)
30. Prove that if D is an exterior point of ΔABC, then there is a line $\overleftrightarrow{DE}$ through D that is contained in the exterior of ΔABC (see Figure 4.35).
31. Suppose that line l meets circle γ in two points C and D. Prove that:
    (a) Point P on l lies inside γ if and only if C * P * D.
    (b) If points A and B are inside γ and on opposite sides of l, then the point E at which AB meets l is between C and D.
32. In Figure 4.36, the pairs of angles ($\measuredangle A'B'B''$, $\measuredangle ABB''$) and ($\measuredangle C'B'B''$, $\measuredangle CBB''$) are called pairs of *corresponding angles* cut off on l and l' by transversal t. Prove that corresponding angles are congruent if and only if alternate interior angles are congruent.
33. *Prove* that there exists a triangle which is not isosceles.

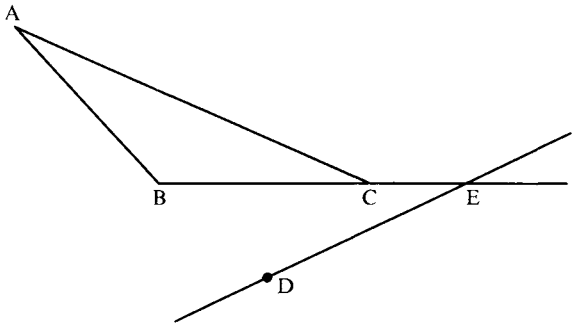

FIGURE 4.35

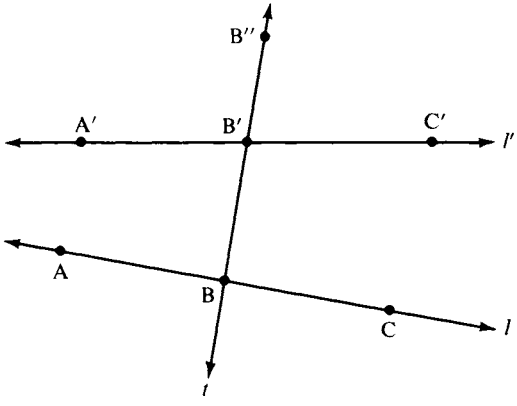

FIGURE 4.36

MAJOR EXERCISES

1. THEOREM. If line l passes through a point A inside circle γ then l intersects γ in two points.

   Here is the idea of the proof; fill in the details using the circular continuity principle (instead of the stronger axiom of Dedekind) and Exercise 27 (see Figure 4.37). Let O be the center of γ. Point B is taken to be the foot of the perpendicular from O to l, point C is taken such that B is the midpoint of OC, and γ' is the circle centered at C having the same radius as γ. Prove that γ' intersects $\overleftrightarrow{OC}$ in a point E′ inside γ and a point E outside γ, so that γ' intersects γ in two points P, P′, and that these points lie on the

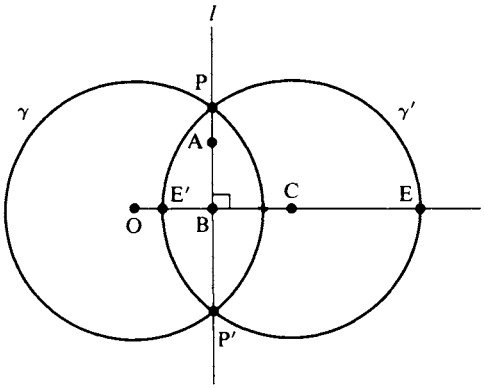

FIGURE 4.37

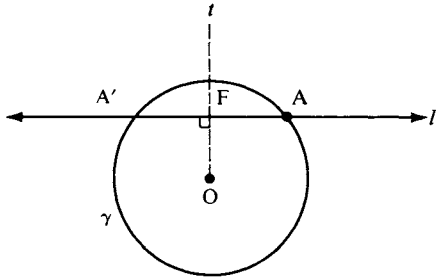

FIGURE 4.38

original line l. (We located the intersections of γ with l by intersecting γ with its *reflection* γ' across l—see p. 111.)

4. Apply the previous exercise to prove that the circular continuity principle implies the elementary continuity principle. (Hint: Use Exercise 27.)
3. Let line l intersect circle γ at point A. If $l \perp \overleftrightarrow{OA}$, where O is the center of γ, we call l *tangent* to γ at A; otherwise l is called *secant* to γ.
   (a) Suppose l is secant to γ. Prove that the foot F of the perpendicular t from O to l lies inside γ and that the reflection A′ of A across t is a second point at which l meets γ. (See Figure 4.38.)
   (b) Suppose now l is tangent to γ. Prove that every point B $\neq$ A lying on l is outside γ, and hence A is the unique point at which l meets γ.
   (c) Let point P lie outside γ. Proposition 7.3, Chapter 7, applies the circular continuity principle to construct a line through P tangent to γ. Explain why that construction is valid only in Euclidean geometry. Prove that the tangent line exists in neutral geometry. (Hint: Let Q $\neq$ P be any point on the perpendicular to $\overleftrightarrow{OP}$ through P. Prove that $\overrightarrow{PQ}$ does not intersect γ whereas $\overrightarrow{PO}$ does. Apply Dedekind's axiom to ray $\overrightarrow{OQ}$. See Figure 4.39.) Once one tangent l through P is obtained, prove that the reflection of l across $\overleftrightarrow{OP}$ is a second one.
4. *Converse to the triangle inequality.* If a, b, and c are lengths of segments such that the sum of any two is greater than the third, then there exists a

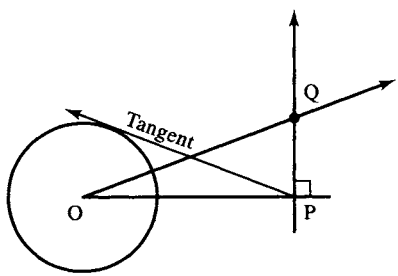

FIGURE 4.39

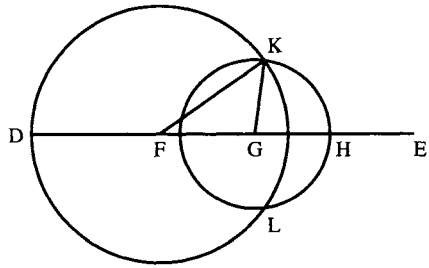

FIGURE 4.40

triangle whose sides have those lengths (Euclid's Proposition 22). Use the circular continuity principle to fill the gap in Euclid's proof and justify the steps: Assume $a \geq b \geq c$. Take any point D and any ray emanating from D. Starting from D, lay off successively on that ray points F, G, H so that $a = \overline{DF}$, $b = \overline{FG}$, $c = \overline{GH}$. Then the circle with center F and radius a meets the circle with center G and radius c at a point K, and ΔFGK is the triangle called for in the proposition. (See Figure 4.40.)

5. Prove that the converse to the triangle inequality implies the circular continuity principle (assuming the incidence, betweenness, and congruence axioms).
6. Prove: If b and c are lengths of segments, then there exists a right triangle with hypotenuse c and leg b if and only if $b < c$. (Hint for the "if" part: Take any point C and any perpendicular lines through C. There exists a point A on one line such that $\overline{AC} = b$. If α is the circle centered at A of radius c, point C lies inside α, and hence α intersects the other line in some point B. Then ΔABC is the requisite right triangle.)
7. Show how the previous exercise furnishes a solution to Major Exercise 3(c) that avoids the use of Dedekind's axiom. (Hint: Let $c = \overline{OP}$ and $b =$ radius of γ and lay off $\measuredangle A$ at O with $\overrightarrow{OP}$ as one side.)
8. Here is an Archimedean proof in neutral geometry of the "important corollary" to Aristotle's axiom, Chapter 3. We must show that given any positive real number a there is a point R on line l such that $(\measuredangle QRP)^\circ < a^\circ$ (intuitively, by taking R sufficiently far out we can get as small an angle as we please). The idea is to construct a sequence of angles $\measuredangle QR_1P$, $\measuredangle QR_2P$, . . . each one of which is at most half the size of its predecessor. Justify the following steps (Figure 4.41):

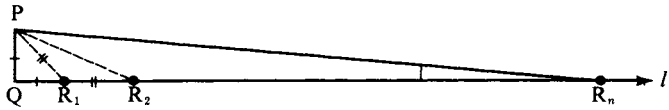

FIGURE 4.41

There exists a point R_1 on l such that $PQ \cong QR_1$ (why?), so that $\triangle PQR_1$ is isosceles. It follows that $(\measuredangle QR_1P)^\circ \leqq 45^\circ$ (why?). Next, there exists a point R_2 such that $Q * R_1 * R_2$ and $PR_1 \cong R_1R_2$, so that $\triangle PR_1R_2$ is isosceles. It follows that $(\measuredangle QR_2P)^\circ \leqq 22\frac{1}{2}^\circ$ (to justify this step, use Corollary 1 to the Saccheri-Legendre theorem). Continuing in this way, we get angles successively less than or equal to $11\frac{1}{4}^\circ$, $5\frac{5}{8}^\circ$, etc. so that by the Archimedean property of real numbers, we eventually get an angle $\measuredangle QR_nP$ with $\measuredangle(QR_nP)^\circ < a^\circ$.

PROJECTS

1. Here is a heuristic argument showing that Archimedes' axiom is necessary to prove the Saccheri-Legendre theorem. It is known that on a sphere, the angle sum of every triangle is greater than 180° (see Kay, 1969); that doesn't contradict the Saccheri-Legendre theorem, because a sphere is not a model of neutral geometry. Fix a point O on a sphere. Consider the set N of all points on the sphere whose distance from O is *infinitesimal.* Interpret "line" to be the arc in N of any great circle. Give "between" its natural interpretation on an arc, and interpret "congruence" as in spherical geometry. Then N becomes a model of our I, B, and C axioms in which Archimedes' axiom and the Saccheri-Legendre theorem do not hold. Similarly, if we fix a point O in a Euclidean plane and take N to be its infinitesimal neighborhood, the angle sum of every triangle is 180°, yet Euclid V does not hold in N (because the point at which the lines are supposed to meet is too far away); thus the converse to Proposition 4.11 cannot be proved from our I, B, and C axioms alone (Aristotle's axiom is needed; see Chapter 5).

   For a rigorous elaboration of this argument, see Hessenberg and Diller (1967) (if you can read German; if you can't, then report on Chapter 32 of Moise, 1990, which constructs a Euclidean ordered field that is not Archimedean).
2. Report on the proof of Theorem 4.3 given in Borsuk and Szmielew, Chapter 3, Sections 9 and 10. The key to the proof is that every Dedekind cut on the ordered set of dyadic rational numbers (see Exercise 18, Chapter 3) determines a unique real number.
3. Our proof of Theorem 4.7 used Archimedes' axiom again. Report on the proof in Martin (1982), Chapter 22, that avoids using this axiom.

4. Given a sphere of radius r, let ϵ be any positive real number $\leq \frac{1}{2}\pi r$ and let N_ϵ be the set of all points on the sphere whose spherical distance from a fixed point O on the sphere is less than ϵ. Interpret "line," "between," and "congruent" as they were interpreted for N in Project 1. Then N_ϵ is not a model of our I, B, and C axioms. Tell which axioms hold and which ones fail. For those that fail, explain heuristically why they hold in N.