CONVENTIONALISM IN GEOMETRY *

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1. Introduction. In what sense and to what extent can the ascription of a particular metric geometry to physical space be held to have an empirical warrant? To answer this question we must inquire whether and how empirical facts function restrictively so as to support a unique metric geometry as the true description of physical space.

The inquiry is prompted by the conflict of ideas on this issue emerging in the Albert Einstein volume in Schilpp’s Library of Living Philosophers between Robertson, Reichenbach and Einstein. Robertson characterizes K. Schwarzschild’s attempt to determine observationally the Gaussian curvature of an astronomical 2-flat as an inspiring implementation of the empiricist conception of physical geometry. And Robertson deems Schwarzschild’s view to be “in refreshing contrast to the pontifical pronouncement of Henri Poincaré,” [25, p. 325] who had declared that “Euclidean geometry has, . . ., nothing to fear from fresh experiments” [20, p. 81] after reviewing the various possible results of stellar parallax measurements. In the same volume [21, p. 297] and elsewhere [22, Ch. 8; 23, pp. 30–37], Reichenbach maintains, as Carnap had done in his early monograph Der Raum [3], that the question as to which metric geometry prevails in physical space is indeed empirical but subject to an important proviso: it becomes empirical only after a physical definition of congruence for line segments has been given conventionally by stipulating (to within a constant factor depending on the choice of unit) what length is to be assigned to a transported solid rod in different positions of space. Reichenbach calls this qualified empiricist conception “the relativity of geometry” and terms “conventionalism” the more radical thesis that even after the physical meaning of “congruent” has been fixed, it is entirely a matter of convention which physical geometry is said to prevail. Believing Poincaré to have been an exponent of conventionalism in this sense, Reichenbach rejects Poincaré’s supposed philosophy of geometry as

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erroneous. On the other hand, Einstein criticizes Reichenbach's relativity of geometry by upholding a particular version of conventionalism which he attributes to Poincaré [9, pp. 676-679].

This exchange reveals that there are several different theses concerning the presence of stipulational ingredients in physical geometry and the warrant for their introduction which require critical examination in the course of our inquiry.

Our main concern is with the respective roles of convention and fact in the ascription of a particular metric geometry to physical space on the basis of measurements with a rigid body. Accordingly, we shall discuss in turn the two principal problems which have been posed in connection with the formulation of the criterion of rigidity and of isochronism.

2. The Criterion of Rigidity: I. The Status of Spatial Congruence. Differential geometry allows us to metrize a given physical surface, say an infinite blackboard or some portion of it, in various ways so as to acquire any metric geometry compatible with its topology. Thus, if we have such a space and a net-work of Cartesian coordinates on it, we can just as legitimately metrize the portion above the x-axis by means of the metric \( ds^2 = \frac{dx^2 + dy^2}{y^2} \), which confers a hyperbolic geometry on that space, as by the Euclidean metric \( ds^2 = dx^2 + dy^2 \). The geometer is not disconcerted by the fact that in the former metrization, the lengths of horizontal segments whose termini have the same coordinate differences \( dx \) will be \( ds = \frac{dx}{y} \) and will thus depend on where they are along the y-axis. What is his sanction for preserving equanimity in the face of the fact that this metrization commits him to regard a segment for which \( dx = 2 \) at \( y = 2 \) as congruent to a segment for which \( dx = 1 \) at \( y = 1 \), although the customary metrization would regard the length ratio of these segments to be \( 2 : 1 \)? His answer would be that unless one of two segments is a subset of the other the congruence of two segments is a matter of convention, stipulation or definition and not a factual matter concerning which empirical findings could show one to have been mistaken. He does not say, of course, that a transported solid rod will coincide successively with the two hyperbolically-congruent segments but allows for this non-coincidence by making the length of the transported rod a suitable function of its position rather than a constant. And in this way, he justifies his claim that the hyperbolic metrization possesses
both epistemological and mathematical credentials as good as those of the Euclidean one.

This conception of congruence was vigorously contested by Bertrand Russell and defended by Poincaré in a controversy which grew out of the publication of Russell's *Foundations of Geometry* [28]. Our first concern will be with the central issue of that debate.

Russell states the factualist's argument as follows [26, pp. 687–688]

> "It seems to be believed that since measurement is necessary to discover equality or inequality, these cannot exist without measurement. Now the proper conclusion is exactly the opposite. Whatever one can discover by means of an operation must exist independently of that operation: America existed before Christopher Columbus, and two quantities of the same kind must be equal or unequal before being measured. Any method of measurement is good or bad according as it yields a result which is true or false. Mr. Poincaré, on the other hand, holds that measurement creates equality and inequality. It follows [then] ... that there is nothing left to measure and that equality and inequality are terms devoid of meaning."

Before setting forth the grounds for regarding Russell's argument here as untenable, it will be useful to analyze the reasoning employed in an inadequate criticism of it. This analysis will exhibit an important facet of the relation of the axiomatic method in pure geometry to the description of physical space.

We are told that Russell's contention can be dismissed by simply pointing to the theory of models: since physical geometry is a semantically-interpreted abstract calculus, the customary physical interpretation of the abstract relation term "congruent" (for line segments) as opposed to the kind of interpretation given in our hyperbolic metrization above clearly cannot itself be a factual statement. Hence it is argued that the alternative metrizability of spatial and temporal continua should never have been either startling or a matter for dispute. On this view, Poincaré could have spared himself the trouble of polemicizing against Russell on behalf of it in the form of a philosophical doctrine of congruence. For, so the argument runs [7, pp. 9–10], there can be nothing particularly problematic about the physical interpretation of the term "congruent": like the physical meaning of all other primitives of the calculus, the denotata of the abstract relation term "congruent" (for line segments) are specified by

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1 An implicit endorsement of this argument is given by H. von Helmholtz [33, p. 15].
semantical rules which are fully on a par in regard to both conventionality
and importance with those furnishing the interpretation of any of the
other abstract primitives of the calculus. In fact, Tarski's axioms for
elementary Euclidean geometry, which appear in this volume, even
dispense with the primitive "congruent" for line segments and yet yield
(the elementary form of) a metric geometry by using instead a quaternary
predicate $\equiv$ denoting the equidistance relation between 4 points.

That such an argument does not go to the heart of the issue and hence
would have failed to convince Russell can be seen from the following:
The congruence relation for line segments, and correspondingly for
regions of surfaces and of 3-space, is a reflexive, symmetrical and tran-
sitive relation in these respective classes of geometrical configurations.
Thus, congruence is a kind of equality relation. Now suppose that one
believes, as Russell and Helmholtz thought they could believe justifiably,
that the spatial equality obtaining between congruent line segments
consists in their each containing the same intrinsic amount of space. Then
one will maintain that in any physico-spatial interpretation of an abstract
geometrical calculus, it is never legitimate to choose arbitrarily what
specific line segments are going to be called "congruent". And, by
the same token, one will assert that in Tarski's aforementioned axio-
omatization, it is never arbitrary what quartets of physical points are to be
regarded as the denotata of his quaternary equidistance predicate $\equiv$.
Instead the imputation of an intrinsic metric to the extended continua of
space and time will issue in the following contentions: (i) since only
"truly equal" intervals may be called "congruent", Newton [18, pp. 6–8]
was right in insisting that there is only one true metrization of the time
continuum, and (ii) there is no room for choice as to the lines which are to
be called "straight" and hence no choice among alternative metric
geometries of physical space, since the geodesic requirement $\frac{\delta f}{ds} = 0$,
which must be satisfied by the straight lines, is imposed subject to the
restriction that only intrinsically congruent line elements may be assigned
the same length $ds$.

These considerations show that it will not suffice in this context simply
to take the model-theoretic conception of geometry for granted and there-
by to dismiss the Russell-Helmholtz claim peremptorily in favor of alter-
native metrizability. Rather what is needed is a refutation of the Russell-
Helmholtz root-assumption of an intrinsic metric: to exhibit the un-
tenability of that assumption is to provide the justification of the model-
theoretic affirmation that a given set of physico-spatial facts may be held
to be as much a realization of a Euclidean calculus as of a non-Euclidean one yielding the same topology.

We shall now see how Riemann and Poincaré furnished the philosophical underpinning for that affirmation.

The following statement in Riemann's Inaugural Dissertation [24, pp. 274, 286] contains a fundamental insight into the particular character of the continuous manifolds of space and time:

"Definite parts of a manifold, which are distinguished from one another by a mark or boundary are called quanta. Their quantitative comparison is effected by means of counting in the case of discrete magnitudes and by measurement in the case of continuous ones. Measurement consists in bringing the magnitudes to be compared into coincidence; for measurement, one therefore needs a means which can be applied (transported) as a standard of magnitude. If it is lacking, then two magnitudes can be compared only if one is a [proper] part of the other and then only according to more or less, not with respect to how much. . . . in the case of a discrete manifold, the principle [criterion] of the metric relations is already implicit in [intrinsic to] the concept of this manifold, whereas in the case of a continuous manifold, it must be brought in from elsewhere [extrinsically]. Thus, either the reality underlying space must form a discrete manifold or the reason for the metric relations must be sought extrinsically in binding forces which act on the manifold." 

Russell [28, pp. 66–67] and the writer [13] have noted that, contrary to Riemann's apparent expectation, the first part of this statement will not bear critical scrutiny as a characterization of continuous manifolds in general. Riemann does, however, render here a fundamental feature of the continua of physical space and time, which are manifolds whose elements, taken singly, all have zero magnitude. And since our concern is with the geo-chronometry of continuous physical space and time, we can disregard defects in his account which do not affect its pertinence to the latter continua. By the same token, we can ignore inadequacies arising from his treatment of discrete and continuous types of order as jointly exhaustive. Instead, we state the valid upshot of his conception relevant to the spatio-temporal congruence issue before us. Construing his statement as applying, not only to lengths but also, mutatis mutandis, to areas and to volumes of higher dimensions, he gives the following

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2 Riemann apparently does not consider sets which are neither discrete nor continuous, but we shall consider the significance of that omission below.
sufficient condition for the intrinsic definability and non-definability of a metric without claiming it to be necessary as well: in the case of a discretely-ordered set, the "distance" between two elements can be defined intrinsically in a rather natural way by the cardinality of the "interval" determined by these elements. On the other hand, upon confronting the extended continuous manifolds of physical space and time, we see that neither the cardinality of intervals nor any of their other topological properties provide a basis for an intrinsically-defined metric. The first part of this conclusion was tellingly emphasized by Cantor's proof of the equi-cardinality of all positive intervals independently of their length. Thus, there is no intrinsic attribute of the space between the end-points of a line-segment AB, or any relation between these two points themselves, in virtue of which the interval AB could be said to contain the same amount of space as the space between the termini of another interval CD not coinciding with AB. Corresponding remarks apply to the time continuum. Accordingly, the continuity we postulate for physical space and time furnishes a sufficient condition for their intrinsic metrical amorphismness.

3 The basis for the discrete ordering is not here at issue: it can be conventional, as in the case of the letters of the alphabet, or it may arise from special properties and relations characterizing the objects possessing the specified order.

4 Clearly, this does not preclude the existence of sufficient conditions other than continuity for the intrinsic metrical amorphismness of sets. But one cannot invoke densely-ordered, denumerable sets of points (instants) in an endeavor to show that discontinuous sets of such elements may likewise lack an intrinsic metric: even without measure theory, ordinary analytic geometry allows the deduction that the length of a denumerably infinite point set is intrinsically zero. This result is evident from the fact that since each point (more accurately, each unit point set or degenerate subinterval) has length zero, we obtain zero as the intrinsic length of the densely-ordered denumerable point set upon summing, in accord with the usual limit definition, the sequence of zero lengths obtainable by denumeration (cf. Grünbaum [11, pp. 297-298]). More generally, the measure of a denumerable point set is always zero (cf. Hobson [15, p. 166]) unless one succeeds in developing a very restrictive intuitionistic measure theory of some sort.

These considerations show incidentally that space-intervals cannot be held to be merely denumerable aggregates. Hence in the context of our post-Cantorian meaning of "continuous", it is actually not as damaging to Riemann's statement as it might seem prima facie that he neglected the denumerable dense sets by incorrectly treating the discrete and continuous types of order as jointly exhaustive. Moreover, since the distinction between denumerable and super-denumerable dense sets was almost certainly unknown to Riemann, it is likely that by "continuous" he merely intended the property which we now call "dense". Evidence of such an earlier usage of "continuous" is found as late as 1914: cf. Russell [27, p. 138].
The axioms of congruence [35, pp. 42-50] preempt "congruent" to be a spatial equality predicate but allow an infinitude of mutually-exclusive congruence classes of intervals. There are no *intrinsic* metric attributes of intervals, however, which could be invoked to single out *one* of these congruence classes as unique. Hence only the *choice* of a particular *extrinsic* congruence standard can determine a unique congruence class, the rigidity of that standard under transport being *decree by convention*. And thus the role of this standard cannot be construed with Russell to be the mere ascertainment of an otherwise intrinsic equality obtaining between the intervals belonging to the congruence class defined by it. Similarly for time intervals and the periodic devices which define temporal congruence. And hence there can be no question at all of an *empirically* or factually determinate metric geometry or chronometry until *after* a physical stipulation of congruence.\(^5\)

A concluding remark on the special importance of the equality term "congruent" (for line segments) vis-à-vis the other primitives of the calculus will precede turning our attention to some of the import of the conventionality of congruence.

Suitable alternative semantical interpretations of the term "congruent", and correlative of "straight line," can readily demonstrate that, subject to the restrictions imposed by the existing topology, it is always a live option to give *either* a Euclidean or a *non*-Euclidean description of the same body of physico-geometrical facts. The possibility of alternative semantical interpretations of such *other* primitives of rival geometrical calculi as "point" does *not* generally have such relevance to this demonstration. Accordingly, when one is concerned, as we are here, with noting that, even apart from the logic of induction, the empirical facts themselves do *not* uniquely dictate the truth of either Euclidean geometry or of one of its non-Euclidean rivals, then the situation is as follows: the different physical interpretations of the term "congruent" (and hence of "straight line") in the respective geometrical calculi enjoy a more central importance in the discussion than the semantics of such other primitives of these calculi as "point," since the latter generally have the *same* physical meaning in both the Euclidean and non-Euclidean descriptions. Moreover, once we cease to look at physical geometry as a descriptively-interpreted system of abstract *synthetic* geometry and regard it instead as an interpreted system of abstract *differential* geometry of the

\(^5\) For a detailed critique of A. N. Whitehead's *perceptualistic* objections to this conclusion [34, ch. VI; 35, ch. III; 36, *passim*] see Grünbaum [13].
Gauss-Riemann type, the pre-eminent status of the interpretation of "congruent" is seen to be beyond dispute: by choosing a particular distance function $ds = \sqrt{g_{ik}dx^idx^k}$ for the line element, we specify not only what segments are congruent and what lines are straights (geodesics) but the entire geometry, since the metric tensor $g_{ik}$ fully determines the Gaussian curvature $K$. To be sure, if one were discussing not the alternative between a Euclidean and non-Euclidean description of the same spatial facts but rather the set of all models (including non-spatial ones) of a given calculus, say the Euclidean one, then indeed the physical interpretation of "congruent" and of "straight line" would not merit any more attention than that of other primitives like "point".

The Import of Riemann's Conception of Congruence.

(a) F. Klein's Relative Consistency Proof of Hyperbolic Geometry and H. Poincaré's *Anschaulichkeitsbeweis* of that geometry.

In the light of the conventionality of congruence, F. Klein's relative consistency proof of hyperbolic geometry via a model furnished by the interior of a circle on the Euclidean plane ⁶ appears as merely one particular kind of possible remetrisation of the circular portion of that plane, projective geometry having played the heuristic role of furnishing Klein with a suitable definition of congruence. What from the point of view of synthetic geometry appears as intertranslatability via a dictionary, appears as alternative metrisability from the point of view of differential geometry. Again, Poincaré's kind of *Anschaulichkeitsbeweis* of a three-dimensional hyperbolic geometry via a model furnished by the interior of a sphere in Euclidean space [20, pp. 75-8] is another example of remetrisation. Here the alteration in our customary definition of congruence is conveyed to us pictorially by the effects of an inhomogeneous force field which appropriately shrinks all bodies alike as seen from the point of view of the normally Euclideanly-behaving bodies.

(b) Poincaré and the Conventionality of Congruence.

The central theme of Poincaré's so called conventionalism is essentially an elaboration of the thesis of alternative metrisability whose fundamental justification we owe to Riemann, and not [12, § 5] the radical conventionalism attributed to him by Reichenbach [23, p. 36].

Poincaré's much-cited and often misunderstood statement concerning the possibility of always giving a Euclidean description of any results of stellar parallax measurements is a less lucid statement of exactly the same point.

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⁶ For details, cf. Bonola [1, pp. 164-175]. For a summary of E. Beltrami's different relative consistency proof, see Struik [31, pp. 152-3].
made by him with magisterial clarity in the following passage [20, p. 235]:

"In space we know rectilinear triangles the sum of whose angles is equal to two right angles; but equally we know curvilinear triangles the sum of whose angles is less than two right angles. . . . To give the name of straights to the sides of the first is to adopt Euclidean geometry; to give the name of straights to the sides of the latter is to adopt the non-Euclidean geometry. So that to ask what geometry it is proper to adopt is to ask, to what line is it proper to give the name straight? It is evident that experiment can not settle such a question."

Now, the equivalence of this contention to Riemann’s view of congruence becomes evident the moment we note that the legitimacy of identifying lines which are curvilinear in the usual geometrical parlance as “straights” is vouchsafed by the warrant for our choosing a new definition of congruence such that the previously curvilinear lines become geodesics of the new congruence. Corresponding remarks apply to Poincaré’s contention that we can always preserve Euclidean geometry in the face of any data obtained from stellar parallax measurements: if the paths of light rays are geodesics on a particular definition of congruence, as indeed they are in the Schwarzschild procedure cited by Robertson, and if the paths of light rays are found parallactically to sustain non-Euclidean relations on that metrization, then we need only choose a different definition of congruence such that these same paths will no longer be geodesics and that the geodesics of the newly chosen congruence are Euclideanly related. From the standpoint of synthetic geometry, the latter choice effects a renaming of optical and other paths and thus is merely a recasting of the same factual content in Euclidean language rather than a revision of the extra-linguistic content of optical and other laws. Since Poincaré’s claim here is a straightforward elaboration of the metric amorphousness of the continuous manifold of space, it is not clear how Robertson can reject it as a “pontifical pronouncement” and even regard it as being in contrast with what he calls Schwarzschild’s “sound operational approach to the problem of physical geometry.” [25, pp. 324–5]. For Schwarzschild had rendered the question concerning the prevailing geometry factual only by the adoption of a particular spatial

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7 The remetrizational retainability of Euclideanism affirmed by Poincaré [20, pp. 81–86] thus involves a merely linguistic interdependence of the geometric theory of rigid solids and the optical theory of light rays. This interdependence is logically different, as we shall see in Section 3, from P. Duhem’s conception [6, Part II, ch. VI] of an epistemological interdependence, which Einstein espouses.
metrization based on the travel times of light, which does indeed turn the
direct light paths of his astronomical triangle into geodesics.

There are two respects, however, in which Poincaré is open to criticism
in this connection:

(i) He maintained [20, p. 81] that it would always be regarded as most
convenient to preserve Euclidean geometry, even at the price of re-
metrization, on the grounds that this geometry is the simplest ana-
lytically [20, p. 65]. Precisely the opposite development materialized in
the general theory of relativity: Einstein forsook the simplicity of the
geometry itself in the interests of being able to maximize the simplicity
of the definition of congruence. He makes clear in his fundamental paper
of 1916 that had he insisted on the retention of Euclidean geometry in a
gravitational field, then he could not have taken “one and the same rod,
independently of its place and orientation, as a realization of the same
interval.” [8, p. 161]

(ii) Even if the simplicity of the geometry itself were the sole determi-
nant of its adoption, that simplicity might be judged by criteria other
than Poincaré’s analytical simplicity. Thus, Menger has urged that
from the point of view of a criterion grounded on the simplicity of the
undefined concepts used, hyperbolic and not Euclidean geometry is the
simplest [16, p. 66].

On the other hand, if Poincaré were alive today, he could point to an
interesting recent illustration of the sacrifice of the simplicity and
accessibility of the congruence standard on the altar of maximum
simplicity of the resulting theory. Astronomers have recently proposed
to remetrize the time continuum for the following reason: when the mean
solar second, which is a very precisely known fraction of the period of
the earth’s rotation on its axis, is used as a standard of temporal con-
gruence, then there are three kinds of discrepancies between the actual
observational findings and those predicted by the usual theory of celestial
mechanics. The empirical facts thus present astronomers with the follow-
ing choice: Either they retain the rather natural standard of temporal
congruence at the cost of having to bring the principles of celestial
mechanics into conformity with observed fact by revising them appropri-
ately. Or they remetrize the time continuum, employing a less simple
definition of congruence so as to preserve these principles intact. Decisions
taken by astronomers in the last few years were exactly the reverse of
Einstein’s choice of 1916 as between the simplicity of the standard of
congruence and that of the resulting theory. The mean solar second is to
be supplanted by a unit to which it is non-linearly related: the sidereal year, which is the period of the earth's revolution around the sun, due account being taken of the irregularities produced by the gravitational influence of the other planets.  

We see that the implementation of the requirement of descriptive simplicity in theory-construction can take alternative forms, because agreement of astronomical theory with the evidence now available is achievable by revising either the definition of temporal congruence or the postulates of celestial mechanics. The existence of this alternative likewise illustrates that for an axiomatized physical theory containing a geochronometry, it is gratuitous to single out the postulates of the theory as having been prompted by empirical findings in contradistinction to deeming the definitions of congruence to be wholly a priori, or vice versa. This conclusion bears out geochronometrically Braithwaite's contention in this volume that there is an important sense in which axiomatized physical theory does not lend itself to compliance with Heinrich Hertz's injunction to "distinguish thoroughly and sharply between the elements ... which arise from the necessities of thought, from experience, and from arbitrary choice." [14, p. 8].

(c) The impossibility of defining congruence uniquely by stipulating a particular metric geometry.

A question which arises naturally upon undertaking the mathematical implementation of a given choice of a metric geometry in the context of a particular set of topological facts is the following: do these facts in conjunction with the desired metric geometry determine a unique definition of congruence? If the answer were actually in the affirmative, as both Carnap [3, pp. 54-55] and Reichenbach [23, pp. 33-34; 22, pp. 132-133] have maintained, this would mean that the desired geometry would uniquely specify a metric tensor under given factual circumstances and thus, in a particular coordinate system, a unique set of functions $g_{ik}$. But Carnap's and Reichenbach's assertion of uniqueness is erroneous, as is demonstrated by showing that besides the customary definition of congruence, which assigns the same length to the measuring rod everywhere and thereby confers a Euclidean geometry on an ordinary table top, there are infinitely many other definitions of congruence which likewise

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8 For a clear account of the relevant astronomical details, see Clemence [4].

9 Braithwaite's point was made independently by Pap [19], who argues that the analytic-synthetic distinction cannot be upheld for partially-interpreted theoretical languages like that of theoretical physics.
yield a Euclidean geometry for that surface but which make the length of a rod depend on its orientation or position. Thus, consider our horizontal table top equipped with a net-work of Cartesian coordinates $x$ and $y$ and suppose that another such surface intersects the horizontal one at an angle $\theta$ so that their line of intersection is both the $y$-axis of the horizontal plane and the $\bar{y}$-axis of a rectangular system of coordinates $\bar{x}$ and $\bar{y}$ on the inclined plane. Assume that the inclined plane has been metrized in the customary way. But then remetrize the horizontal plane by calling congruent in it those line segments which are the perpendicular projections onto it of segments of the inclined plane that are equal in the latter's metric. Accordingly, we have a mapping

$$\bar{x} = x \sec \theta$$
$$\bar{y} = y,$$

and we now assign to a line segment of the horizontal plane whose termini have the coordinate differences $dx$ and $dy$ not the customary length $\sqrt{dx^2 + dy^2}$ but rather

$$ds = \sqrt{\bar{x}^2 + \bar{y}^2} = \sqrt{\sec^2 \theta dx^2 + dy^2}.$$

Nonetheless, upon using the new $g_{ik}$, which are introduced into the $x$, $y$ coordinates by the revised definition of congruence, to compute the Gaussian curvature of the horizontal table top, we still obtain the Euclidean value zero. And by merely varying the angle of inclination $\theta$, we obtain infinitely many different definitions of congruence all of which make the length of a given rod dependent on its orientation and yet impart a Euclidean geometry to the horizontal table top. Thus, the requirement of Euclideanism does not uniquely determine a metric tensor, and, contrary to Carnap and Reichenbach, there are infinitely many ways in which a measuring rod could squirm under transport as compared to its customary behavior and still yield a Euclidean geometry. In fact, even for plane Euclidean geometry, the class of congruence definitions is far wider than the one-parameter family yielded by our particular isometric mappings of an inclined plane onto the horizontal one. Dr. Samuel Gulden, to whom I presented the problem of determining the class of different metric tensors for each kind of two-dimensional and three-dimensional Riemannian space, has pointed out that (i) in the Euclidean case, upon abandoning the restriction of our above isometric mappings to affine coordinate transformations and considering non-linear transformations with non-vanishing Jacobian, we can generate infinitely many other
metrizations whose associated Gaussian curvature is everywhere zero. For example, for the admissible transformation between our two sets of rectangular coordinates \( x, y \) and \( \tilde{x}, \tilde{y} \) given by

\[
\tilde{x} = x + \frac{1}{3}y^3, \quad \text{and} \\
\tilde{y} = \frac{1}{3}x^3 - y,
\]

the distance function becomes

\[
ds^2 = d\tilde{x}^2 + d\tilde{y}^2 = (1 + x^4)dx^2 + 2(y^2 - x^2)dxdy + (y^4 + 1)dy^2.
\]

In this case, the length of a given rod is generally dependent both on its position and on its orientation, (ii) the result obtained for Euclidean space can be generalized to a very large class of Riemann spaces of various dimensions.

We are now ready to consider the second of the two principal problems which have been posed in connection with the criterion of rigidity.

3. The Criterion of Rigidity: II. The Logic of Correcting for “Distorting” Influences. Physical geometry is usually conceived as the system of metric relations exhibited by transported solid bodies independently of their particular chemical composition. On this conception, the criterion of congruence can be furnished by a transported solid body for the purpose of determining the geometry by measurement, only if the computational application of suitable “corrections” (or, ideally, appropriate shielding) has essentially eliminated inhomogeneous thermal, elastic, electric and other influences, which produce changes of varying degree (“distortions”) in different kinds of materials. The demand for this elimination as a prerequisite to the experimental determination of the geometry has a thermodynamic counterpart: the requirement of a means for measuring temperature which does not yield the discordant results produced by expansion thermometers at other than fixed points when different thermometric substances are employed. This thermometric need is fulfilled successfully by Kelvin’s thermodynamic scale of temperature. But attention to the implementation of the corresponding prerequisite of physical geometry has led Einstein [9, pp. 676–678] to impugn the empirical status of that geometry. He considers the case in which congruence has been defined by the diverse kinds of transported solid measuring rods as corrected for their respective idiosyncratic distortions with a view to then making an empirical determination of the prevailing geometry. And in an
argument which he attributes to Poincaré, Einstein's thesis is that the very logic of computing these corrections precludes that the geometry itself be accessible to experimental ascertainment in isolation from other physical regularities. Specifically, he states the case in the form of a dialogue between Reichenbach and Poincaré 10:

"Poincaré: The empirically given bodies are not rigid, and consequently can not be used for the embodiment of geometric intervals. Therefore, the theorems of geometry are not verifiable.

Reichenbach: I admit that there are no bodies which can be immediately adduced for the "real definition" of the interval. Nevertheless, this real definition can be achieved by taking the thermal volume-dependence, elasticity, electro- and magneto-striction, etc., into consideration. That this is really [and] without contradiction possible, classical physics has surely demonstrated.

Poincaré: In gaining the real definition improved by yourself you have made use of physical laws, the formulation of which presupposes (in this case) Euclidean geometry. The verification, of which you have spoken, refers, therefore, not merely to geometry but to the entire system of physical laws which constitute its foundation. An examination of geometry by itself is consequently not thinkable. — Why should it consequently not be entirely up to me to choose geometry according to my own convenience (i.e., Euclidean) and to fit the remaining (in the usual sense "physical") laws to this choice in such manner that there can arise no contradiction of the whole with experience?"

The objection which Einstein presents here on behalf of conventionalism is aimed at a conception of physical geometry which is empiricist merely in Carnap's and Reichenbach's conditional sense explained in Section 1. Einstein's criticism is that the rigid body is not even defined without first decreeing the validity of Euclidean geometry. And the grounds he gives for this conclusion are that before the corrected rod can be used to make an empirical determination of the de facto geometry, the required corrections must be computed via laws, such as those of elasticity, which involve Euclideanly-calculated areas and volumes. But clearly the warrant

10 It is rather doubtful that Poincaré himself espoused the version of conventionalist which Einstein links to his name here: in speaking of the variations which solids exhibit under distorting influences, Poincaré says [20, p. 76]: "we neglect these variations in laying the foundations of geometry, because, besides their being very slight, they are irregular and consequently seem to us accidental."
for thus introducing Euclidean geometry *at this stage* cannot be empirical.

I now wish to set forth my reasons for believing that Einstein's argument does not succeed in making physical geometry a matter of convention rather than fact in a sense which is *independent* of the alternative metrizability vouchsafed by spatio-temporal continuity.

There is no question that the laws used to make the corrections for deformations [30, p. 60; 32, p. 408] involve areas and volumes in a fundamental way (e.g. in the definitions of the elastic stresses and strains) and that this involvement presupposes a geometry, as is evident from the area and volume formulae

\[
A = \int \sqrt{g} \, dx^1 dx^2 \quad \text{and} \quad V = \int \sqrt{g} \, dx^1 dx^2 dx^3,
\]

where "g" represents the determinant of the components \( g_{ik} \) [10, p. 177]. Now suppose that we begin with a set of Euclideanly-formulated physical laws \( P_0 \) in correcting for the distortions induced by perturbations and then use the thus Euclideanly-corrected congruence standard for *empirically* exploring the geometry of space by determining the metric tensor. The initial stipulational affirmation of the Euclidean geometry \( G_0 \) in the physical laws \( P_0 \) used to compute the corrections in no way assures that the geometry obtained by the corrected rods will be Euclidean! If it is non-Euclidean, then the question is: what will Einstein's fitting of the physical laws to preserve Euclideanism and avoid a contradiction of the total theoretical system with experience involve? Will the adjustments in \( P_0 \) necessitated by the retention of Euclideanism entail merely a change in the dependence of the length assigned to the transported rod on such non-positional parameters as temperature, pressure, magnetic field etc.? Or could the putative empirical findings compel that the length of the transported rod be likewise made a function of its position and orientation in order to square the coincidence findings with the requirement of Euclideanism? The temporal variability of distorting influences and the possibility of obtaining non-Euclidean results by measurements carried out in a spatial region uniformly characterized by standard conditions of temperature, pressure, electric and magnetic field strength etc. show it to be quite doubtful that the preservation of Euclideanism could always be accomplished short of introducing the dependence of the rod's length on position and orientation. Thus, the need for *remetrizing* in this sense in order to retain Euclideanism cannot be ruled out. But this kind of remetrization does not provide the requisite support for Einstein's version of conventionalism, whose onus it is to show that the geometry by itself
cannot be held to be empirical even when we exclude resorting to such remetrization.

That the geometry may well be empirical in this sense is seen from the following possibilities of its successful empirical determination. After assumably obtaining a non-Euclidean geometry \( G_1 \) from measurements with a rod corrected on the basis of Euclideanly-formulated physical laws \( P_0 \), we can revise \( P_0 \) so as to conform to the non-Euclidean geometry \( G_1 \) just obtained by measurement. This retroactive revision of \( P_0 \) would be effected by recalculating such quantities as areas and volumes on the basis of \( G_1 \) and changing the functional dependencies relating them to temperature and other physical parameters. We thus obtain a new set of laws \( P_1 \). Now we use this set \( P_1 \) of laws to correct the rods for perturbational influences and then determine the geometry with the thus corrected rods. If the result is a geometry \( G_2 \) different from \( G_1 \), then if there is convergence to a geometry of constant curvature, we must repeat this process a finite number of times until the geometry \( G_n \) ingredient in the laws \( P_n \) providing the basis for perturbation-corrections is indeed the same to within experimental accuracy as the geometry obtained by measurements with rods that have been corrected via the set \( P_n \).

If there is such convergence at all, it will be to the same geometry \( G_n \) even if the physical laws used in making the initial corrections are not the set \( P_0 \), which presupposes Euclidean geometry, but a different set \( P \) based on some non-Euclidean geometry or other. That there can exist only one such geometry of constant curvature \( G_n \) would seem to be guaranteed by the identity of \( G_n \) with the unique underlying geometry \( G_t \) characterized by the following properties: (i) \( G_t \) would be exhibited by the coincidence behavior of a transported rod if the whole of the space were actually free of deforming influences, (ii) \( G_t \) would be obtained by measurements with rods corrected for distortions on the basis of physical laws \( P_t \) presupposing \( G_t \), and (iii) \( G_t \) would be found to prevail in a given relatively small, perturbation-free region of the space quite independently of the assumed geometry ingredient in the correctional physical laws. Hence, if our method of successive approximation does converge to a geometry \( G_n \) of constant curvature, then \( G_n \) would be this unique underlying geometry \( G_t \). And, in that event, we can claim to have found empirically that \( G_t \) is indeed the geometry prevailing in the entire space which we have explored.

But what if there is no convergence? It might happen that whereas convergence would obtain by starting out with corrections based on the
set $P_0$ of physical laws, it would not obtain by beginning instead with corrections presupposing some particular non-Euclidean set $P$ or vice versa: just as in the case of Newton’s method of successive approximation [5, p. 286], there are conditions, as A. Suna has pointed out to me, under which there would be no convergence. We might then nonetheless succeed as follows in finding the geometry $G_t$ empirically, if our space is one of constant curvature.

The geometry $G_r$ resulting from measurements by means of a corrected rod is a single-valued function of the geometry $G_a$ assumed in the correctional physical laws, and a Laplacian demon having sufficient knowledge of the facts of the world would know this function $G_r = f(G_a)$. Accordingly, we can formulate the problem of determining the geometry empirically as the problem of finding the point of intersection between the curve representing this function and the straight line $G_r = G_a$. That there exists one and only one such point of intersection follows from the existence of the geometry $G_t$ defined above, provided that our space is one of constant curvature. Thus, what is now needed is to make determinations of the $G_r$ corresponding to a number of geometrically-different sets of correctional physical laws $P_a$, to draw the most reasonable curve $G_r = f(G_a)$ through this finite number of points $(G_a, G_r)$, and then to find the point of intersection of this curve and the straight line $G_r = G_a$.

Whether this point of intersection turns out to be the one representing Euclidean geometry or not is beyond the reach of our conventions, barring a remetrization. And thus the least that we can conclude is that since empirical findings can greatly narrow down the range of uncertainty as to the prevailing geometry, there is no assurance of the latitude for the choice of a geometry which Einstein takes for granted. Einstein’s Duhemian position would appear to be inescapable only if our proposed method of determining the geometry by itself empirically cannot be generalized in some way to cover the general relativity case of a space of variable curvature and if the latter kind of theory turns out to be true.

It would seem therefore that, contrary to Einstein, the logic of eliminating distorting influences prior to stipulating the rigidity of a solid body is not such as to provide scope for the ingestion of conventions over and above those acknowledged in Riemann’s analysis of congruence, and trivial ones such as the system of units used. Our analysis of the logical status of the concept of a rigid body thus leads to the conclusion that once the physical meaning of congruence has been stipulated by reference to a solid body for whose distortions allowance has been made compu-
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tationally as outlined, then the geometry is determined uniquely by the
totality of relevant empirical facts. It is true, of course, that even apart
from experimental errors, not to speak of quantum limitations on the
accuracy with which the metric tensor of space-time can be meaningfully
ascertained by measurement [29; 37], no finite number of data can uniquely
determine the functions constituting the representations $g_{ik}$ of the
metric tensor in any given coordinate system. But the criterion of inductive
simplicity which governs the free creativity of the geometer's imagination
in his choice of a particular metric tensor here is the same as the one
employed in theory formation in any of the non-geometrical portions of
empirical science. And choices made on the basis of such inductive
simplicity are in principle true or false, unlike those springing from
considerations of descriptive simplicity, which merely reflect conventions.

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